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Equivalence classes of homotopy-associative comultiplications of finite complexes

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Abstract

Let X be a finite, 1-connected CW-complex which admits a homotopy-associative comultiplication. Then X has the rational homology of a wedge of spheres, $S^{n_1+1} \vee \cdots \vee S^{n_r+1}$. Two comultiplications of X are equivalent if there is a self-homotopy equivalence of X which carries one to the other. Let $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$, respectively $\tilde{\mathscr{C}}_{\mathrm{ac}}(X)$, denote the set of equivalence classes of homotopy classes of homotopy-associative, respectively, homotopy-associative and homotopy-commutative, comultiplications of X. We prove the following basic finiteness result: Theorem 6.1 (1) If for each i, (a) $n_i \neq n_j + n_k$ for every j, k with $j < k$ and (b) $n_i \neq 2n_j$ for every j with n_j even, then $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$ is finite. (2) $\tilde{\mathscr{C}}_{\mathrm{ac}}(X)$ is always finite. The methods of proof are algebraic and consist of a detailed examination of comultiplications of the free Lie algebra $\pi_{\#}(\Omega X) \otimes \mathbb{Q}$. These algebraic methods and results appear to be of interest in their own right. For example, they provide dual versions of well-known results about Hopf algebras. In an appendix we show the group of self-homotopy equivalences that induce the identity on all homology groups is finitely generated.

1. Introduction

In this paper we consider various sets of comultiplications of a topological space, up to a suitable notion of equivalence. A *comultiplication of a space* X is a map $\alpha: X \to X \vee X$ such that $p\alpha$ and $p'\alpha$ are both homotopic to the identity map of X, where p and p' are the projections $X \vee X \to X$ onto the first and second summands of the wedge. Examples of comultiplications abound, and a large and important class occurs when X is a suspension and α is the natural pinching map. Two comultiplications α and β of a space X are *equivalent* if there is a homotopy equivalence $f: X \to X$ such that βf and $(f \vee f)\alpha: X \to X \vee X$ are homotopic. There are natural definitions for a comultiplication α to be homotopy-associative or homotopy-commutative

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which we recall below. These properties are preserved under equivalence and we investigate equivalence classes of comultiplications with these properties. In particular, we consider basic finiteness questions such as the following: For which spaces are there finitely many equivalence classes of homotopy-associative comultiplications? For which spaces are there finitely many equivalence classes of homotopy-associative and homotopy-commutative comultiplications? We give rather complete answers to these questions.

In the dual context, multiplications of a space X have been extensively studied, and it is known that the possibilities are limited. Curjel has shown that any finite complex admits finitely many equivalence classes of homotopy-associative multiplications [9, Theorem I]. The questions concerning comultiplications that we examine here are dual to those in [9]. In general, however, the results for equivalence classes of comultiplications of finite complexes are more diverse than for equivalence classes of multiplications of finite complexes. For instance, $S^3 \vee S^5$ admits infinitely many equivalence classes of homotopy-associative comultiplications (Example 6.6(1)).

Let $\mathscr{C}(X)$ denote the set of homotopy classes of comultiplications of X, and $\mathscr{C}_{\mathrm{a}}(X)$, respectively $\mathscr{C}_{\mathrm{ac}}(X)$, the set of homotopy classes of homotopy-associative comultiplications, respectively of both homotopy-associative and homotopy-commutative comultiplications, of X. We denote the equivalence classes by $\tilde{\mathscr{C}}(X)$, $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$ and $\tilde{\mathscr{C}}_{\mathrm{ac}}(X)$. Our main results concern a finite complex X which admits a homotopy-associative comultiplication. It is known that X and a wedge of spheres $S^{n_1+1} \vee \cdots \vee S^{n_r+1}$ have the same rational homotopy type. We prove in Theorem 6.1 that if for each i, (a) $n_i \neq n_j + n_k$ for every j, k with $j < k$ and (b) $n_i \neq 2n_j$ for every j with n_j even, then $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$ is finite. Also in Theorem 6.1, we prove that $\tilde{\mathscr{C}}_{\mathrm{ac}}(X)$ is always finite.

The paper is organized as follows: In this section we give definitions and fix notation. In Section 2 we establish basic facts concerning comultiplications and rationalization. The main result is Proposition 2.3 which enables us to obtain information about comultiplications of a space X by studying comultiplications of the rational homotopy Lie algebra $\pi_{\#}(\Omega X)\otimes\mathbb{Q}$ of X. The latter is a purely algebraic and very effective context in which to work. In Section 3 we relate $\tilde{\mathscr{C}}(X)$ to equivalence classes of comultiplications of the corresponding rational homotopy Lie algebra $\pi_{\#}(\Omega X)\otimes\mathbb{Q}$. The main result here is Theorem 3.14, which essentially allows us to conclude that $\tilde{\mathscr{C}}(X)$, $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$ or $\tilde{\mathscr{C}}_{\mathrm{ac}}(X)$ are finite whenever the corresponding equivalence classes of Lie algebra comultiplications of $\pi_{\#}(\Omega X)\otimes\mathbb{Q}$ are finite. In Sections 4 and 5 we work in an algebraic setting and show that certain Lie algebras admit only finitely many equivalence classes of comultiplications. These results are applied in Section 6, but some of them are interesting in their own right. Theorem 4.4 asserts that an associative comultiplication on a Lie algebra is determined up to equivalence by its quadratic part. Proposition 5.3 and Corollary 5.5 are dual to well-known results about Hopf algebras. The main result, Theorem 6.1, then follows easily from the previous sections. Also in Section 6 we give several examples that illustrate our results. In an appendix we prove that, for a finite complex, the group of self-homotopy equivalences that induce the identity on homology groups is finitely generated. Our work uses this result, but it also seems of independent interest and is not available in the literature.

We end this section with conventions and notations that are adopted and used throughout the paper. A topological space will either be a 1-connected, based space of the homotopy type of a based CW complex or the rationalization of such a space. All maps and homotopies are to preserve basepoints. We do not distinguish notationally between a map and its homotopy class. However, we sometimes signify homotopy of maps by $\simeq$ and same homotopy type of spaces by $\equiv$. For spaces X and Y, we let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. If X is a space, then the graded homotopy group of X is denoted by $\pi_{\#}(X)$.

We also consider graded Lie algebras over the rationals $\mathbb{Q}$, referred to as Lie algebras, and homomorphisms of Lie algebras. The *rational homotopy Lie algebra* of a space X is $\pi_{\#}(\Omega X)\otimes\mathbb{Q}$, where ΩX is the loop-space of X and the bracket operation is obtained from the Samelson product [23, Ch.X, Section 5]. We often denote this Lie algebra by L_X. If $f: X \to Y$, then $f_{\#}: L_X = \pi_{\#}(\Omega X)\otimes\mathbb{Q} \to L_Y = \pi_{\#}(\Omega Y)\otimes\mathbb{Q}$ denotes the induced homomorphism of Lie algebras. A Lie algebra L is *free* if there is a graded $\mathbb{Q}$-vector space V with $L \cong \mathbb{L}(V)$, the free Lie algebra generated by V (see [21, p. 16]). All Lie algebras in this paper are free Lie algebras. In addition, we only consider the case where V is positively graded and finite-dimensional, so that each V_n is a finite-dimensional vector space with $V_n = 0$ for $n \leq 0$ and for n sufficiently large. If $\{y_1, \dots, y_r\}$ is a graded basis for V, then we write $\mathbb{L}(V) = \mathbb{L}(y_1, \dots, y_r)$, and $\{y_1, \dots, y_r\}$ is called a *basis* or a *set of generators* for the Lie algebra $\mathbb{L}(V)$. We say $x \in \mathbb{L}(y_1, \dots, y_r)$ if $x \in \mathbb{L}(y_1, \dots, y_r)_n$ for some n, and write $|x|$ for the degree n of x. If $x \in \mathbb{L}(y_1, \dots, y_r)$, then x has *length* k if x can be expressed as a linear combination of brackets of length k in the generators $y_1, \dots, y_r$. A *Hall basis* for $\mathbb{L}(V)$ is a totally ordered, graded vector space basis for the underlying graded vector space of $\mathbb{L}(V)$. Such a basis is constructed as follows: Choose a totally ordered basis $y_1 < \cdots < y_r$ for V. Then the remaining basis elements for $\mathbb{L}(V)$ are certain monomials in the y_j's, using the bracket operation as product, which are chosen and ordered by proceeding inductively over bracket length. Once the elements of length $\leq s$ have been chosen and ordered, this determines the basis elements of length $s+1$ according to certain rules (see [11] or [20, LA, Section 4.5)]). The length $s+1$ basis elements are then ordered arbitrarily amongst themselves, and each is given greater order than any element of length $\leq s$. We remark that in the graded case, a Hall basis for $\mathbb{L}(V)$ includes not only the 'basic products' of [11], but also the squares of basic products of odd degree (see [18, 4.5]).

The *coproduct* of Lie algebras L and L' is denoted $L \sqcup L'$ and defined as follows: If $L = \mathbb{L}(V)$ and $L' = \mathbb{L}(V')$, then $L \sqcup L' = \mathbb{L}(V \oplus V')$. A (*Lie algebra*) *comultiplication* of L is a homomorphism $\phi: L \to L \sqcup L$ such that $\pi\phi$ and $\pi'\phi$ both equal the identity homomorphism of L, where π and π' are the projections $L \sqcup L \to L$ onto the first and second summands (see [4, Section 2]). Elements of $L \sqcup L$ in either summand are distinguished from each other by using a prime on elements from the second summand. Thus, if $L = \mathbb{L}(y_1, \dots, y_r)$, then $L \sqcup L = \mathbb{L}(y_1, \dots, y_r, y_1', \dots, y_r')$.

Some of our proofs in Section 3 use the universal enveloping algebra functor U from the category of Lie algebras to the category of associative algebras (see [19, Appendix B] or [21] for details). In particular, a Lie algebra $\mathbb{L}(V)$ has universal enveloping algebra $U(\mathbb{L}(V)) = T(V)$, the tensor algebra on V, with inclusion $i: \mathbb{L}(V) \to T(V)$ the unique Lie algebra homomorphism that extends the identity on V. The *coproduct* of tensor algebras $T(V)$ and $T(V')$ is denoted $T(V) \sqcup T(V')$ and defined by $T(V) \sqcup T(V') = T(V \oplus V')$. A *comultiplication* of $T(V)$ is a homomorphism $\psi: T(V) \to T(V) \sqcup T(V)$ such that $\pi\psi$ and $\pi'\psi$ both equal the identity homomorphism of $T(V)$, where π and π' are the projections $T(V) \sqcup T(V) \to T(V)$ onto the first and second summands (see [7]). By applying the universal enveloping algebra functor to a Lie algebra comultiplication $\phi: \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V)$, one obtains a comultiplication $U(\phi): T(V) \to T(V) \sqcup T(V)$.

If $\alpha: X \to X \vee X$ is a comultiplication of X, then the pair (X, α) is called a *co-H-space*. Often we omit reference to the comultiplication and call the space X a co-H-space. If X is a co-H-space which is a finite CW complex, then it is known [6, Theorem 2.2] that X has the same rational homotopy type as a wedge of spheres $S^{n_1+1} \vee \cdots \vee S^{n_r+1}$, $n_1 \leq \cdots \leq n_r$. It follows that the rational homotopy Lie algebra $L_X = \pi_{\#}(\Omega X) \otimes \mathbb{Q}$ is a free Lie algebra $\mathbb{L}(y_1, \ldots, y_r)$ with $|y_i| = n_i$. The sequence of integers $(n_1, \ldots, n_r)$ is called the *type* of the co-H-space X. Note that the type is determined either by L_X or by the rational homology $H_*(X; \mathbb{Q})$. A comultiplication $\alpha: X \to X \vee X$ which satisfies $(1 \vee \alpha)\alpha \simeq (\alpha \vee 1)\alpha: X \to X \vee X \vee X$ is called *homotopy-associative* and the co-H-space X is also called homotopy-associative. By [13, Theorem 2.3] a homotopy-associative co-H-space always has homotopy inverses and is therefore a *cogroup*. Thus the term cogroup will mean homotopy-associative co-H-space. A *finite cogroup* is a cogroup which is a finite CW complex. A prime example of a finite cogroup is the suspension of a finite complex with the natural pinching map as comultiplication. A comultiplication $\alpha: X \to X \vee X$ is called *homotopy-commutative* if $T\alpha \simeq \alpha: X \to X \vee X$, where $T: X \vee X \to X \vee X$ is the switching map which interchanges coordinates. Many of these terms carry over to a multiplication ϕ of a Lie algebra L. We call ϕ *associative* if $(1 \sqcup \phi)\phi = (\phi \sqcup 1)\phi: L \to L \sqcup L \sqcup L$ and we call ϕ *commutative* if $T\phi = \phi: L \to L \sqcup L$, where $T: L \sqcup L \to L \sqcup L$ is the switching homomorphism which interchanges summands.

Let $\mathscr{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of a space X and $\mathscr{E}_*(X)$ the subgroup of $\mathscr{E}(X)$ of homotopy equivalences that induce the identity on all homology groups. Let $\operatorname{Aut} L$ denote the group of automorphisms of a Lie algebra L and $\operatorname{Aut}_* L$ the subgroup of automorphisms that induce the identity on the vector space of indecomposables $L/[L, L]$. The group $\mathscr{E}(X)$ acts on $\mathscr{C}(X)$ as follows: If $f \in \mathscr{E}(X)$ and $\alpha \in \mathscr{C}(X)$, then $f * \alpha$ is the composition

$$X \xrightarrow{f^{-1}} X \xrightarrow{\alpha} X \vee X \xrightarrow{f \vee f} X \vee X,$$

which is a comultiplication of X. The set of orbits under this action are the equivalence classes $\tilde{\mathscr{C}}(X)$ of comultiplications. Clearly homotopy-associativity and homotopy-commutativity are preserved by this action and so $\mathscr{E}(X)$ also acts on $\mathscr{C}_a(X)$ and $\mathscr{C}_{ac}(X)$. The corresponding sets of orbits are the sets of equivalence classes $\tilde{\mathscr{C}}_a(X)$ and $\tilde{\mathscr{C}}_{ac}(X)$. Likewise, for a Lie algebra L, $\operatorname{Aut} L$ acts on $\mathscr{C}(L)$, the set of comultiplications on L: If $\theta \in \operatorname{Aut} L$ and $\phi \in \mathscr{C}(L)$, then $\theta * \phi = (\theta \sqcup \theta)\phi\theta^{-1}$. Writing the set of associative comultiplications on L as $\mathscr{C}_a(L)$ and the set of associative and commutative comultiplications as $\mathscr{C}_{ac}(L)$, we have the corresponding sets of equivalence classes denoted $\tilde{\mathscr{C}}_a(L)$ and $\tilde{\mathscr{C}}_{ac}(L)$.

In addition, we often consider the action of the subgroup $\mathscr{E}_*(X)$ on the sets $\mathscr{C}(X)$, $\mathscr{C}_a(X)$ and $\mathscr{C}_{ac}(X)$, and the subgroup $\operatorname{Aut}_* L$ on the sets $\mathscr{C}(L)$, $\mathscr{C}_a(L)$ and $\mathscr{C}_{ac}(L)$. Our reasons for doing so are explained in Remark 3.15. In these cases we use the notation $S//G$ to denote the set of orbits or equivalence classes under the action of a group G on a set S. We call a function $f\colon A \to B$ of sets *finite-to-one* if the inverse image of every point of B is a finite set.

For a Lie algebra L, $\mathscr{C}(L)$ and $\operatorname{Aut}_* L$ are conveniently described in terms of a given set of generators. If $L = \mathbb{L}(y_1, \dots, y_r)$, then as above $L \sqcup L = \mathbb{L}(y_1, \dots, y_r, y'_1, \dots, y'_r)$. A homomorphism $\phi\colon L \to L \sqcup L$ is a comultiplication if and only if, for each y_j,

$$\phi(y_j) = y_j + y'_j + \sum_{s \geq 2} P_s(y_j), \tag{1.1}$$

where $P_s(y_j)$ is a polynomial of length s in $y_1, \dots, y_r, y'_1, \dots, y'_r$, each monomial of which contains at least one entry from $y_1, \dots, y_r$ and at least one entry from $y'_1, \dots, y'_r$. Here, and in what follows, polynomials in a Lie algebra are obtained by using the bracket operation as multiplication. We call P, defined by $P(y_j) = \sum_{s \geq 2} P_s(y_j)$, the *perturbation* of the comultiplication ϕ[4, p. 84]. Similarly, an automorphism $\theta\colon L \to L$ is in $\operatorname{Aut}_* L$ if and only if, for each generator y_j,

$$\theta(y_j) = y_j + \sum_{s \geq 2} Q_s(y_j), \tag{1.2}$$

where $Q_s(y_j)$ is a polynomial of length s in $y_1, \dots, y_r$.

2. Rationalization and comultiplications

We review some relevant facts concerning rationalization or $\mathbb{Q}$-localization of groups and spaces; for more details, see [12]. A group G is called $\mathbb{Q}$-*local* if for every positive integer n, the map $x \to x^n$ is a bijection $G \to G$. A homomorphism $f\colon G \to H$ of groups is called a $\mathbb{Q}$-*isomorphism* if (1) the kernel of f is a torsion group and (2) for every $x \in H$, there is a positive integer n such that x^n is in the image of f. Next, a topological space Y is called a *rational space* if $\pi_i(Y)$ is $\mathbb{Q}$-local for every i. For any space X, there exists the *rationalization* of X, written $X_{\mathbb{Q}}$, which is a rational space, and the rationalization map $l\colon X \to X_{\mathbb{Q}}$ which is universal with respect to maps of X into rational spaces. Spaces X and Y are said to have the *same rational homotopy type* if $X_{\mathbb{Q}} \equiv Y_{\mathbb{Q}}$. The rationalization construction is functorial so that $\gamma \in [X, Y]$

yields $\gamma_{\mathbb{Q}} \in [X_{\mathbb{Q}}, Y_{\mathbb{Q}}]$. Thus we have a *rationalization function* $e: [X, Y] \to [X_{\mathbb{Q}}, Y_{\mathbb{Q}}]$, defined by $e(\gamma) = \gamma_{\mathbb{Q}}$. One easily sees that the rationalization function restricts to functions $e: \mathscr{C}(X) \to \mathscr{C}(X_{\mathbb{Q}})$, $e: \mathscr{C}_a(X) \to \mathscr{C}_a(X_{\mathbb{Q}})$, $e: \mathscr{E}(X) \to \mathscr{E}(X_{\mathbb{Q}})$, $e: \mathscr{E}_*(X) \to \mathscr{E}_*(X_{\mathbb{Q}})$ and so on.

We now recall some rational homotopy theory; for details see [21]. A basic result asserts that each rational space X has a *Quillen model*, i.e., a differential graded Lie algebra minimal model $L(X)$. Furthermore, there is a notion of homotopy for homomorphisms between Quillen models. The Quillen model is functorial and for rational spaces X and Y it gives a bijection between homotopy classes $[X, Y]$ and (differential graded Lie algebra) homotopy classes $[L(X), L(Y)]$. Next, consider a co-H-space X. Since the rational homotopy Lie algebra $L_X = \pi_\#(\Omega X) \otimes \mathbb{Q}$ is a free Lie algebra, it follows that the Quillen model of $X_{\mathbb{Q}}$ is just L_X with zero differential. But if X is a co-H-space, then so is $X \vee X$, and the Quillen model of $(X \vee X)_{\mathbb{Q}}$ is again just the rational homotopy Lie algebra $L_{X \vee X} = \pi_\#(\Omega(X \vee X)) \otimes \mathbb{Q} = L_X \sqcup L_X$ with zero differential. Now for homomorphisms between Quillen models that have zero differentials, homotopy reduces to equality. So for co-H-spaces X and Y, $[L(X_{\mathbb{Q}}), L(Y_{\mathbb{Q}})] = \operatorname{Hom}(L_X, L_Y)$. Hence the set of Lie algebra homomorphisms $\operatorname{Hom}(L_X, L_X \sqcup L_X)$ is in one-one correspondence with the set of homotopy classes of maps $[X_{\mathbb{Q}}, (X \vee X)_{\mathbb{Q}}]$. Moreover, this correspondence restricts to give bijections between $\mathscr{C}(L_X)$ and $\mathscr{C}(X_{\mathbb{Q}})$, $\mathscr{C}_a(L_X)$ and $\mathscr{C}_a(X_{\mathbb{Q}})$, and $\mathscr{C}_{ac}(L_X)$ and $\mathscr{C}_{ac}(X_{\mathbb{Q}})$ (see [4, Section 2]). In a similar way, if X is a co-H-space, then we identify $\operatorname{Aut} L_X$ with $\mathscr{E}(X_{\mathbb{Q}})$ and $\operatorname{Aut}_* L_X$ with $\mathscr{E}_*(X_{\mathbb{Q}})$.

If $\alpha \in \mathscr{C}(X)$, then the induced homomorphism $\alpha_\# : \pi_\#(\Omega X) \otimes \mathbb{Q} \to \pi_\#(\Omega(X \vee X)) \otimes \mathbb{Q}$ is a Lie algebra comultiplication of L_X. Thus there is a function g: $\mathscr{C}(X) \to \mathscr{C}(L_X)$ defined by $g(\alpha) = \alpha_\#$. By the above discussion, this function can be identified with the restriction of the rationalization function e: $\mathscr{C}(X) \to \mathscr{C}(X_{\mathbb{Q}})$. Similarly, an element $f \in \mathscr{E}(X)$ induces $f_\# \in \operatorname{Aut} L_X$, and if $f \in \mathscr{E}_*(X)$, then $f_\# \in \operatorname{Aut}_* L_X$. This gives a homomorphism $h: \mathscr{E}(X) \to \operatorname{Aut} L_X$ that restricts to h: $\mathscr{E}_*(X) \to \operatorname{Aut}_* L_X$. Again, by the above discussion, we identify this function with the restriction of the rationalization function e: $\mathscr{E}_*(X) \to \mathscr{E}_*(X_{\mathbb{Q}})$.

We begin by interpreting $\mathscr{C}(X)$ in terms of homotopy sets. If X is a cogroup with comultiplication α, then it is well-known that α induces a group structure – denoted multiplicatively – on the set $[X, Y]$, for any space Y. Now let $X \flat X$ be the space of paths in $X \times X$ which begin in $X \vee X$ and end at the basepoint of $X \times X$ and let $j: X \flat X \to X \vee X$ be the map that projects a path onto its initial point. In other words, $X \flat X$ is the homotopy-fibre of the inclusion $X \vee X \to X \times X$.

2.1. Lemma. *Let X be a finite cogroup. The function Ψ: $[X, X \flat X] \to \mathscr{C}(X)$ defined by $\Psi(\beta) = \alpha \cdot (j\beta)$ is a bijection, where α is the given comultiplication of X.*

We omit the proof of Lemma 2.1; the dual case is well-known (see, for example, [1, Lemma 2]).

2.2. Lemma. (cf. [12, Corollary 6.5] and [8, V, Proposition 5.3]). *If X is a finite cogroup and Y is a space such that $\pi_i(Y)$ is finitely generated for all i, then $[X, Y]$ is a finitely generated nilpotent group and the homomorphism*

$$l_* : [X, Y] \to [X, Y_{\mathbb{Q}}],$$

induced by the $\mathbb{Q}$-localization map l: $Y \to Y_{\mathbb{Q}}$, is a $\mathbb{Q}$-isomorphism.

Proof. Let $Y^{(n)}$ denote the nth Postnikov section of Y and let p: $\Sigma\Omega X \to X$ be the map defined by $p(f, t) = f(t)$ for $f \in \Omega X$ and $0 \le t \le 1$. Since X is a cogroup, it is well-known that there is a map s: $X \to \Sigma\Omega X$ such that s is a co-H-map, i.e., is compatible with the comultiplications on X and $\Sigma\Omega X$, and such that $ps = 1$[10, Theorem 2.2]. Thus for some n we have the commutative diagram

$$\begin{tikzcd}
{[X,Y]} \arrow[r, phantom, "\simeq"] \arrow[d, "l_*"] & {[X,Y^{(n)}]} \arrow[r, shift left, "p^*"] \arrow[d, "l_*^{(n)}"] & {[\Sigma\Omega X, Y^{(n)}]} \arrow[l, shift left, "s^*"] \arrow[r, phantom, "\cong"] \arrow[d, "l_*^{(n)}"] & {[\Omega X, \Omega Y^{(n)}]} \arrow[d, "(\Omega l^{(n)})_*"] \\
{[X,Y_{\mathbb{Q}}]} \arrow[r, phantom, "\cong"] & {[X,Y_{\mathbb{Q}}^{(n)}]} \arrow[r, shift left, "p^*"] & {[\Sigma\Omega X, Y_{\mathbb{Q}}^{(n)}]} \arrow[l, shift left, "s^*"] \arrow[r, phantom, "\cong"] & {[\Omega X, \Omega Y_{\mathbb{Q}}^{(n)}]}
\end{tikzcd}$$

with p^* a one-to-one function and s^* an epimorphism.

One proves, as in [3, p. 16], by induction on n that $[\Omega X, \Omega Y^{(n)}]$ is a finitely generated nilpotent group. Observe that nilpotency also follows from [23, X, Corollary 3.8] because $[\Omega X, \Omega Y^{(n)}] \cong [(\Omega X)^N, \Omega Y^{(n)}]$ for some skeleton $(\Omega X)^N$ of ΩX. Since s^* is an epimorphism, $[X, Y]$ is a finitely generated nilpotent group. To complete the proof it suffices to show that $(\Omega l^{(n)})_*$ is a $\mathbb{Q}$-isomorphism. Note that $\Omega l^{(n)}$ is just the $\mathbb{Q}$-localization map l: $\Omega Y^{(n)} \to \Omega Y_{\mathbb{Q}}^{(n)}$. Therefore, we need only prove that $l_* : [\Omega X, \Omega Y^{(n)}] \to [\Omega X, \Omega Y_{\mathbb{Q}}^{(n)}]$ is a $\mathbb{Q}$-isomorphism. For this, one establishes by induction on n that $l_* : [A, \Omega Y^{(n)}] \to [A, \Omega Y_{\mathbb{Q}}^{(n)}]$ is a $\mathbb{Q}$-isomorphism for all $n \ge 2$ and all connected – but not necessarily 1-connected – spaces A such that $H_i(A)$ is finitely generated for all i. □

We now use the bijection Ψ of Lemma 2.1 to transfer group structure to the set $\mathscr{C}(X)$.

2.3. Proposition. *Let X be a finite complex.*

(1) *$\mathscr{E}_*(X)$ is a finitely generated group and h: $\mathscr{E}_*(X) \to \mathrm{Aut}_* L_X$ is a $\mathbb{Q}$-isomorphism.*

(2) *If X is a cogroup, then the sets $\mathscr{C}(X)$ and $\mathscr{C}(L_X)$ can be given group structure such that $\mathscr{C}(X)$ is finitely generated and g: $\mathscr{C}(X) \to \mathscr{C}(L_X)$ is a $\mathbb{Q}$-isomorphism. Furthermore, the kernel of g is finite.*

Proof (1) As in the above discussion, we identify $\mathrm{Aut}_* L_X$ with $\mathscr{E}_*(X_{\mathbb{Q}})$ and h: $\mathscr{E}_*(X) \to \mathrm{Aut}_* L_X$ with the rationalization function e: $\mathscr{E}_*(X) \to \mathscr{E}_*(X_{\mathbb{Q}})$. By the Appendix, $\mathscr{E}_*(X)$ is a finitely generated group. By [16], e: $\mathscr{E}_*(X) \to \mathscr{E}_*(X_{\mathbb{Q}})$ is a $\mathbb{Q}$-isomorphism – in fact, the rationalizing homomorphism.

(2) We use bijections Ψ and Ψ' as in Lemma 2.1 and note that the following diagram commutes:

$$\begin{tikzcd} \mathscr{C}(X) \arrow[d, "g"] & & [X, X \flat X] \arrow[ll, "\Psi"'] \arrow[d, "e"] \\ \mathscr{C}(L_X) \equiv \mathscr{C}(X_{\mathbb{Q}}) & & [X_{\mathbb{Q}}, X_{\mathbb{Q}} \flat X_{\mathbb{Q}}] \arrow[ll, "\Psi'"'] \end{tikzcd}$$

where e is the rationalization function. Since e is the composition

$$[X, X \flat X] \xrightarrow{l_*} [X, (X \flat X)_{\mathbb{Q}}, (X \flat X)_{\mathbb{Q}}] \cong [X_{\mathbb{Q}}, X_{\mathbb{Q}} \flat X_{\mathbb{Q}}],$$

where $l\colon X \flat X \to (X \flat X)_{\mathbb{Q}}$ is the $\mathbb{Q}$-localization map, it follows from Lemma 2.2 that e, and hence g, is a $\mathbb{Q}$-isomorphism.

Next, $[X, X \flat X]$ is a finitely generated nilpotent group by Lemma 2.2. Since e: $[X, X \flat X] \to [X_{\mathbb{Q}}, X_{\mathbb{Q}} \flat X_{\mathbb{Q}}]$ is a $\mathbb{Q}$-isomorphism, the kernel of e is a torsion subgroup of $[X, X \flat X]$. But every torsion subgroup of a finitely generated nilpotent group is finite [(15, p. 232] and [3, Section 2]). Thus the kernel of g: $\mathscr{C}(X) \to \mathscr{C}(L_X)$ is finite. $\square$

Now let $L = \mathbb{L}(y_1, \ldots, y_r)$ and denote by $\operatorname{Int} \mathbb{L}(y_1, \ldots, y_r)$ the subset of L consisting of all elements which can be written as a polynomial in $y_1, \ldots, y_r$ with integer coefficients. We remark that an element is in $\operatorname{Int} \mathbb{L}(y_1, \ldots, y_r)$ if, and only if, when it is written as a linear combination of elements from a Hall basis constructed from $\{y_1, \ldots, y_r\}$, the coefficients are integers. An element $\phi \in \mathscr{C}(L)$ is called *integral* (with respect to the generators $y_1, \ldots, y_r$) if $\phi(y_i) \in \operatorname{Int} \mathbb{L}(y_1, \ldots, y_r, y'_1, \ldots, y'_r)$ for each $i = 1, \ldots, r$. Let $\operatorname{Int} \mathscr{C}(\mathbb{L}(y_1, \ldots, y_r))$ denote the set of integral Lie algebra comultiplications. Similarly, an element $\theta \in \operatorname{Aut} L$ is called *integral* (with respect to $y_1, \ldots, y_r$) if $\theta(y_i) \in \operatorname{Int} \mathbb{L}(y_1, \ldots, y_r)$ for each i. Then $\operatorname{Int} \operatorname{Aut} \mathbb{L}(y_1, \ldots, y_r)$ denotes the set of integral automorphisms and $\operatorname{Int} \operatorname{Aut}_* \mathbb{L}(y_1, \ldots, y_r)$ denotes the integral automorphisms in $\operatorname{Aut}_* L$.

2.4. Lemma. *If X is a finite cogroup, then there is a set of generators $x_1, \ldots, x_r$ for the Lie algebra $L_X = \pi_\#(\Omega X) \otimes \mathbb{Q}$ such that*:

(1) $\operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$ *is a subgroup of $\mathscr{C}(L_X)$ with respect to the group structure of Proposition* 2.3, *and* $\operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \ldots, x_r)$ *is a subgroup of* $\operatorname{Aut}_* L_X$;

(2) *If g: $\mathscr{C}(X) \to \mathscr{C}(L_X)$ and h: $\mathscr{E}_*(X) \to \operatorname{Aut}_* L_X$ are as defined above, then* $\operatorname{Image} g \subseteq \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$ *and* $\operatorname{Image} h \subseteq \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \ldots, x_r)$.

Proof. By Proposition 2.3, there exist finite sets of generators $\alpha_1, \ldots, \alpha_a$ of the group $\mathscr{C}(X)$ and $f_1, \ldots, f_b$ of the group $\mathscr{E}_*(X)$. Let $\alpha_0 \in \mathscr{C}(X)$ be the cogroup comultiplication of X and let $f_0\colon X \to X$ be the canonical inverse map with respect to α_0. We claim that there exists a basis $x_1, \ldots, x_r$ of L_X such that $\alpha_{i\#}$ and $f_{j\#}$ are integral with respect to this basis, $i = 0, \ldots, a$ and $j = 0, \ldots, b$. By applying the claim to α_0 and f_0 we see that $\operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$ is a subgroup of $\mathscr{C}(L_X)$. Lemma 2.4 then follows. To prove the claim let $y_1, \dots, y_r$ be any basis of L_X. We show by induction that for every $s = 1, \dots, r$, there exists a basis $x_1, \dots, x_s, y_{s+1}, \dots, y_r$ of L_X such that all $\alpha_{i\#}(x_k)$ and $f_{j\#}(x_k)$ are integral and furthermore that $x_k = N_k y_k$ for positive integers N_k, for each $k = 1, \dots, s$. Let $\varepsilon_j = -1$ for $j = 0$ and $\varepsilon_j = 1$ for $j > 0$. Since $\alpha_{i\#}(y_1) = y_1 + y'_1$ and $f_{j\#}(y_1) = \varepsilon_j y_1$, the result holds for $s = 1$ with $x_1 = y_1$. Now suppose that the inductive hypothesis holds for s. We write, for each i and j,

$$\alpha_{i\#}(y_{s+1}) = y_{s+1} + y'_{s+1} + \sum_I \frac{a_{(i,I)}}{b_{(i,I)}} Y_I,$$

$$f_{j\#}(y_{s+1}) = \varepsilon_j y_{s+1} + \sum_J \frac{a'_{(j,J)}}{b'_{(j,J)}} Y_J,$$

where each Y_I is a monomial of length ≥ 2 in the generators $y_1, \dots, y_s, y'_1, \dots, y'_s$, each Y_J is a monomial of length ≥ 2 in the generators $y_1, \dots, y_s$, and $a_{(i,I)}$, $b_{(i,I)}$, $a'_{(j,J)}$ and $b'_{(j,J)}$ are integers. Since each y_k that appears in Y_I or Y_J is $(1/N_k)x_k$ and each y'_k that appears in Y_I is $(1/N_k)x'_k$, we have

$$\alpha_{i\#}(y_{s+1}) = y_{s+1} + y'_{s+1} + \sum_I \frac{a_{(i,I)}}{c_{(i,I)}} X_I,$$

$$f_{j\#}(y_{s+1}) = \varepsilon_j y_{s+1} + \sum_J \frac{a'_{(j,J)}}{c'_{(j,J)}} X_J,$$

for some integers $c_{(i,I)}$ and $c'_{(j,J)}$, where each X_I is a monomial of length ≥ 2 in $x_1, \dots, x_s, x'_1, \dots, x'_s$ and each X_J is a monomial of length ≥ 2 in $x_1, \dots, x_s$. We set

$$N_{s+1} = \left(\prod_{i,I} c_{(i,I)}\right)\left(\prod_{j,J} c'_{(j,J)}\right).$$

Then we have

$$\alpha_{i\#}(N_{s+1} y_{s+1}) = N_{s+1} y_{s+1} + N_{s+1} y'_{s+1} + \sum_I d_{(i,I)} X_I,$$

$$f_{j\#}(N_{s+1} y_{s+1}) = \varepsilon_j N_{s+1} y_{s+1} + \sum_J d'_{(j,J)} X_J$$

for some integers $d_{(i,I)}$ and $d'_{(j,J)}$. Set $x_{s+1} = N_{s+1} y_{s+1}$ to complete the induction. The claim then follows by taking $s = r$. □

2.5. Remarks. (1) In the rest of the paper we use the generators $x_1, \dots, x_r$ of L_X constructed in Lemma 2.4 without explicit mention, and so the conclusion of Lemma 2.4 will hold.

(2) Although Lemma 2.4 is dual to Lemma 5.2. of [9], the proof that we have given is not dual to Curjel's proof. We have used localization methods applied to the groups $\mathscr{E}_*(X)$ and $\mathscr{C}(X)$, which were not available earlier. We have also avoided the dual of the condition that the rational cohomology algebra be primitively generated, which was used in Curjel's argument.

As a consequence of Lemma 2.4, the homomorphisms g and h induce homomorphisms

$$g{:}\mathscr{C}(X) \to \mathrm{Int}\mathscr{C}(\mathbb{L}(x_1, \dots, x_r)) \quad \text{and} \quad h{:}\ \mathscr{E}_*(X) \to \mathrm{Int}\mathrm{Aut}_*\mathbb{L}(x_1, \dots, x_r). \tag{2.3}$$

The following is now an immediate consequence of Proposition 2.3.

2.6. Corollary. *If X is a finite cogroup, then the homomorphism g has finite kernel and the homomorphism h has the property that for every $\theta \in \mathrm{Int}\mathrm{Aut}_*\mathbb{L}(x_1, \dots, x_r)$, there is a positive integer n such that $\theta^n \in \mathrm{Image}\, h$.*

3. Reduction to Lie algebras

In this section we relate the set $\mathscr{C}(X)//\mathscr{E}_*(X)$ to the set $\mathscr{C}(L_X)//\mathrm{Aut}_*L_X$, in such a way so as to reduce a study of the former to a study of the latter. This is done by first relating the set $\mathscr{C}(X)//\mathscr{E}_*(X)$ to the set $\mathrm{Int}\mathscr{C}(L_X)//\mathrm{Int}\mathrm{Aut}_*L_X$ (Proposition 3.4) and then relating the set $\mathrm{Int}\mathscr{C}(L_X)//\mathrm{Int}\mathrm{Aut}_*L_X$ to the set $\mathscr{C}(L_X)//\mathrm{Aut}_*L_X$ (Proposition 3.12).

Most of the proofs in this section proceed by induction over a given set of generators for a Lie algebra L. We fix a set of generators $\{x_1, \dots, x_r\}$ of $\mathbb{L}(V)$ with $|x_i| = n_i$ and $n_1 \le \cdots \le n_r$. In the case that X is a finite cogroup, $L = L_X = \pi_\#(\Omega X) \otimes \mathbb{Q}$ and the generators are chosen as in Lemma 2.4.

Let $\mathscr{H}$ denoted a Hall basis for L, constructed from $x_1, \dots, x_r$. Let $\hat{\mathscr{H}} = \mathscr{H} - \{x, \dots, x_r\}$, so that $\hat{\mathscr{H}}$ consists of the decomposable elements of $\mathscr{H}$. We write $\hat{\mathscr{H}} = \{X_1, X_2, \dots\}$.

3.1. Definition. An automorphism $\psi \in \mathrm{Aut}_*L$ is an *elementary automorphism* of L if, for some $m \in \{1, \dots, r\}$, we have

$$\psi(x_i\} = \begin{cases} x_m + X_j & \text{for } i = m, \\ x_i & \text{for } i \ne m \end{cases}$$

for some $X_j \in \hat{\mathscr{H}}$, where $|x_m| = |X_j|$. Note that ψ is in $\mathrm{Int}\mathrm{Aut}_*L$.

For every positive integer n, let El^n be the subgroup of $\mathrm{Int}\mathrm{Aut}_*L$ generated by the set of nth powers of elementary automorphisms of L.

3.2. Lemma. *For every n*, El^n has finite index in $\mathrm{Int}\mathrm{Aut}_*\mathbb{L}(x_1, \dots, x_r)$.

Proof. We describe a finite set of coset representatives for $\mathrm{Int}\mathrm{Aut}_*L/\mathrm{El}^n$. Let $\theta \in \mathrm{In}$-tAut_*L and assume inductively that there exists $\psi_{(k)} \in \mathrm{El}^n$ satisfying, for each $i \le k$,

$$(\psi_{(k)} \circ \theta)(x_i) = x_i + \sum_j a^i_j X_j$$

with a_j^i integers such that $0 \le a_j^i \le n-1$, where the sum is over all j with $X_j \in \hat{\mathscr{H}}$ and $|X_j| = |x_i|$. Then on the next generator,

$$(\psi_{(k)} \circ \theta)(x_{k+1}) = x_{k+1} + \sum_j (a_j^{k+1} + nb_j^{k+1}) X_j,$$

for integers a_j^{k+1} and b_j^{k+1} with $0 \le a_j^{k+1} \le n-1$. Now consider the elementary automorphisms ψ_j defined by $\psi_j(x_{k+1}) = x_{k+1} + X_j$ for each j in the latter sum. A straightforward calculation shows that for $i \le k+1$,

$$\left(\left(\prod_j \psi_j^{-nb_j^{k+1}}\right) \circ \psi_{(k)} \circ \theta\right)(x_i) = x_i + \sum_j a_j^i X_j.$$

We set

$$\psi_{(k+1)} = \left(\prod_j \psi_j^{-nb_j^{k+1}}\right) \circ \psi_{(k)} = \left(\prod_j (\psi_j^n)^{-b_j^{k+1}}\right) \circ \psi_{(k)}$$

to complete the induction.

Therefore for any $\theta \in \operatorname{Int}\operatorname{Aut}_* L$, the coset of θ in $\operatorname{Int}\operatorname{Aut}_* L/\mathrm{El}^n$ contains some $\tilde{\theta} \in \operatorname{Int}\operatorname{Aut}_* L$ given by

$$\tilde{\theta}(x_i) = x_i + \sum_j a_j^i X_j,$$

where $0 \le a_j^i \le n-1$. Since the indexing sets for j are finite, for each $i = 1, \dots, r$, there are finitely many such $\tilde{\theta}$. □

3.3. Lemma. *Let X be a finite cogroup and let h: $\mathscr{E}_*(X) \to \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ be the homomorphism in* (2.3). *Then* Image h *has finite index in* $\operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$.

Proof. By Lemma 3.2, it suffices to show $\mathrm{El}^N \subseteq \operatorname{Image} h$ for some N. Let ψ_j^i be the elementary automorphism defined by $\psi_j^i(x_i) = x_i + X_j$ for $X_j \in \hat{\mathscr{H}}$. By Corollary 2.6 there is some positive integer $n(i,j)$ such that $(\psi_j^i)^{n(i,j)} \in \operatorname{Image} h$. There are finitely many elementary automorphisms ψ_j^i and we let N be the product of all the $n(i,j)$. Therefore $(\psi_j^i)^N \in \operatorname{Image} h$ for all i, j. Hence $\mathrm{El}^N \subseteq \operatorname{Image} h$ since El^N is generated by the $(\psi_j^i)^N$. □

The homomorphisms $g\colon \mathscr{C}(X) \to \operatorname{Int}\mathscr{C}(L_X)$ and h: $\mathscr{E}_*(X) \to \operatorname{Int}\operatorname{Aut}_* L_X$, as in (2.3), induce a map

$$p\colon \mathscr{C}(X)//\mathscr{E}_*(X) \to \operatorname{Int}\mathscr{C}(L_X)//\operatorname{Int}\operatorname{Aut}_* L_X. \tag{3.4}$$

3.4 Proposition. *The map p is finite-to-one.*

Proof. For $\alpha \in \mathscr{C}(X)$, we write $\langle\alpha\rangle$ for its equivalence class in $\mathscr{C}(X)//\mathscr{E}_*(X)$, and for $\phi \in \operatorname{Int}\mathscr{C}(L_X)$ we write $[\phi]$ for its equivalence class in $\operatorname{Int}\mathscr{C}(L_X)//\operatorname{Int}\operatorname{Aut}_* L_X$. By Lemma 3.3, there are finitely many cosets in $\operatorname{Int}\operatorname{Aut}_* L_X/\operatorname{Image} h$, so let $\{\theta_1, \dots, \theta_s\}$

be a set of coset representatives. Then any $\theta \in \operatorname{Int}\operatorname{Aut}_* L_X$ can be written $\theta = f_\# \circ \theta_j$ for some $j = 1, \dots, s$ and some $f \in \mathscr{E}_*(X)$. Now let $[\phi] \in \operatorname{Int}\mathscr{C}(L_X)//\operatorname{Int}\operatorname{Aut}_* L_X$ and consider the set $\{\theta_1 * \phi, \dots, \theta_s * \phi\} \subseteq \operatorname{Int}\mathscr{C}(L_X)$. For each $j = 1, \dots, s$, let $\{\alpha_1^j, \dots, \alpha_{m_j}^j\} \subseteq \mathscr{C}(X)$ be all comultiplications which induce $\theta_j * \phi$, i.e., $\{\alpha_1^j, \dots, \alpha_{m_j}^j\} = g^{-1}(\theta_j * \phi)$. By Corollary 2.6 this set is finite. We now show that $p^{-1}[\phi] = \{\langle\alpha_1^1\rangle, \dots, \langle\alpha_{m_1}^1\rangle, \dots, \langle\alpha_1^s\rangle, \dots, \langle\alpha_{m_s}^s\rangle\}$. Let $\langle\alpha\rangle \in \mathscr{C}(X)//\mathscr{E}_*(X)$ be such that $p\langle\alpha\rangle = [\phi]$. Then there exists $\theta \in \operatorname{Int}\operatorname{Aut}_* L_X$ with $\alpha_\# = \theta * \phi$. But $\theta = f_\# \circ \theta_j$, for some $j = 1, \dots, s$ and some $f \in \mathscr{E}_*(X)$, and so $\alpha_\# = (f_\# \circ \theta_j) * \phi = f_\# * (\theta_j * \phi)$. Therefore $(f^{-1} * \alpha)_\# = f_\#^{-1} * \alpha_\# = \theta_j * \phi$. By construction, $f^{-1} * \alpha = \alpha_i^j$ for some i. Thus $\langle\alpha\rangle = \langle\alpha_i^j\rangle$ and the proposition is proved. $\square$

We now relate the orbit sets $\operatorname{Int}\mathscr{C}(L_X)//\operatorname{Int}\operatorname{Aut}_* L_X$ and $\mathscr{C}(L_X)//\operatorname{Aut}_* L_X$.

3.5. Definition. Let s be an integer ≥ 2 and let $\mathbb{Q}(s) = \{p/q \in \mathbb{Q}\colon q \geq 2,\ q \mid s! \text{ and } 0 \leq p < q\}$. If $\theta \in \operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ we write $\theta(x_i) = x_i + \sum_{s \geq 2} Q_s(x_i)$, as in (1.2). We define $\mathscr{F} \subset \operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ as follows: $\theta \in \mathscr{F}$ if and only if for each i and s, $Q_s(x_i)$ is a polynomial in $x_1, \dots, x_r$ with coefficients from $\mathbb{Q}(s)$.

3.6 Remark. For a given i and s, there are finitely many possibilities for $Q_s(x_i)$ in Definition 3.5, if θ is in $\mathscr{F}$. It follows that $\mathscr{F}$ is a finite set of automorphisms.

Let $\theta \in \mathscr{F}$ and write the length s part of $\theta(x_i)$ as

$$Q_s(x_i) = \sum_j c_{s,j}^i Y_{s,j} \tag{3.5}$$

for $c_{s,j}^i \in \mathbb{Q}(s)$, where the sum is over all brackets $Y_{s,j}$ of length s and degree $|x_i|$.

3.7. Lemma. *Let $\theta \in \operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ with $\theta(x_i) = x_i + \sum_{s \geq 2} Q_s(x_i)$. If, for each i and s, $sQ_s(x_i) \in \operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$, then there exists $\tilde{\theta} \in \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ such that $\tilde{\theta} \circ \theta \in \mathscr{F}$.*

Proof. Since $sQ_s(x_i) \in \operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$ we write $Q_s(x_i) = R_s(x_i) + T_s(x_i)$, with $T_s(x_i) \in \operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$ and

$$R_s(x_i) = \sum_j c_{s,j}^i Y_{s,j},$$

where $c_{s,j}^i \in \mathbb{Q}(s)$ and $Y_{s,j}$ has length s and degree $|x_i|$. We 'remove' the integral part $T_s(x_i)$, proceeding inductively over the generators as in Lemma 3.2.

Assume inductively that there exists $\theta_k \in \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$, such that $(\theta_k \circ \theta)(x_i)$ has length s part as in (3.5), for all $s \geq 2$ and $i \leq k$. Suppose

$$\theta(x_{k+1}) = x_{k+1} + \sum_{s \geq 2} (R_s(x_{k+1}) + T_s(x_{k+1}))$$

as above. Then

$$(\theta_k \circ \theta)(x_{k+1}) = \theta_k(x_{k+1}) + \sum_{s \geq 2} (\theta_k \circ R_s)(x_{k+1}) + \sum_{s \geq 2} (\theta_k \circ T_s)(x_{k+1}). \tag{3.6}$$

By assumption $\theta_k \in \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$, so $\theta_k(x_{k+1})$ and each $(\theta_k \circ T_s)(x_{k+1})$ are in $\operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$. Also, $(\theta_k \circ R_s)(x_{k+1}) = \sum_j c_{s,j}^{k+1} \theta_k(Y_{s,j})$, where $c_{s,j}^{k+1} \in \mathbb{Q}(s)$. Since $\theta_k \in \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$, each $\theta_k(Y_{s,j}) \in \operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$ and is an integral linear combination of brackets of length $\geq s$. We collect together the terms of homogeneous length t that are contributed by the summands $(\theta_k \circ R_s)(x_{k+1})$ for $s \leq t$. For each $s \leq t$, the denominators of the coefficients in $R_s(x_{k+1})$ divide $s!$. Thus the length t part of $(\theta_k \circ R_s)(x_{k+1})$ can be written as a sum of terms from $\operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$ plus a sum of terms having rational coefficients whose denominators divide $s!$ and hence $t!$. It follows that

$$\sum_{s \geq 2} (\theta_k \circ R_s)(x_{k+1}) = \sum_{t \geq 2} \sum_j a_{t,j}^{k+1} Y_{t,j} + \sum_{s \geq 2} U_s^{k+1},$$

with $U_s^{k+1} \in \operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$ of length s and $a_{t,j}^{k+1} \in \mathbb{Q}(t)$. We rewrite (3.6) as

$$(\theta_k \circ \theta)(x_{k+1}) = x_{k+1} + \sum_{t \geq 2} \sum_j a_{t,j}^{k+1} Y_{t,j} + \sum_{s \geq 2} V_s(x_{k+1}),$$

with $V_s(x_{k+1}) \in \operatorname{Int}\mathbb{L}(x_1, \dots, x_r)$, and $a_{t,j}^{k+1} \in \mathbb{Q}(t)$. Define $\psi \in \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ by

$$\psi(x_{k+1}) = x_{k+1} - \sum_{s \geq 2} V_s(x_{k+1}),$$

and $\psi(x_i) = x_i$ for $i \neq k+1$. Then $(\psi \circ \theta_k \circ \theta)(x_i) = (\theta_k \circ \theta)(x_i)$ for $i = 1, \dots, k$ and $(\psi \circ \theta_k \circ \theta)(x_{k+1}) = x_{k+1} + \sum_t \sum_j a_{t,j}^{k+1} Y_{t,j}$. The inductive step is completed by setting $\theta_{k+1} = \psi \circ \theta_k$. Induction starts with $k = 1$, where we take $\theta_1 = 1$. The lemma follows by setting $\tilde{\theta} = \theta_r$. $\square$

We now prove some technical results concerning the universal enveloping algebra $T(V)$ of a Lie algebra $\mathbb{L}(V)$. We use $\mathscr{C}(T)$ to denote the set of comultiplications of a tensor algebra T (see Section 1). We denote by $\operatorname{Aut}_* T$ the automorphisms of T which induce the identity on the indecomposables. If $\{x_1, \dots, x_r\}$ is a basis for V, then we write $T(V) = T(x_1, \dots, x_r)$, and with respect to this choice of generators, $\operatorname{Int} T(x_1, \dots, x_r)$ denotes the subalgebra of $T(V)$ consisting of all polynomials in $x_1, \dots, x_r$ that have integer coefficients. Then $\operatorname{Int}\mathscr{C}(T(x_1, \dots, x_r))$ and $\operatorname{Int}\operatorname{Aut}_* T(x_1, \dots, x_r)$ are defined in analogy to $\operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \dots, x_r))$ and $\operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ in Section 2. In addition, the following notation will be used: Let $T(x_1', \dots, x_r')$, with $|x_i'| = n_i$ and $T(x_1'', \dots, x_r'')$, with $|x_i''| = n_i$, be copies of $T = T(V)$. Then $T \sqcup T \sqcup T = T(x_1, \dots, x_r, x_1', \dots, x_r', x_1'', \dots, x_r'')$.

3.8. Notation. Let $\phi_{(1)}: \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V)$ be the comultiplication defined by $\phi_{(1)}(v) = v + v'$ for $v \in V$. Let $\delta: \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V)$ be the homomorphism defined by $\delta(v) = v'$ for $v \in V$. Applying the universal enveloping algebra functor U, we obtain a comultiplication $U(\phi_{(1)}): T(V) \to T(V) \sqcup T(V)$ and a homomorphism $U(\delta)$: $T(V) \to T(V) \sqcup T(V)$. In terms of a basis $\{x_1, \dots, x_r\}$ for V we have $U(\phi_{(1)})(x_j) = x_j + x'_j$ and $U(\delta)(x_j) = x'_j$.

3.9. Lemma. *Let $\xi \in T(x_1, \dots, x_r)$ be of length at least two. If $U(\phi_{(1)})(\xi) - \xi - U(\delta)(\xi)$ is in $\operatorname{Int} T(x_1, \dots, x_r, x'_1, \dots, x'_r)$, then ξ is in $\operatorname{Int} T(x_1, \dots, x_r)$.*

Proof. First we show that, for ξ of any length, if $U(\phi_{(1)})(\xi) - \xi \in \operatorname{Int} T(x_1, \dots, x_r, x'_1, \dots, x'_r)$, then $\xi \in \operatorname{Int} T(x_1, \dots, x_r)$. We proceed by induction over the length of ξ. In the length 1 case, this is clearly true. Now suppose that the result is true for length $m - 1$ and suppose that ξ has length m. Write

$$\xi = \sum_{i,J} a_{iJ} x_i x_J,$$

where $J = \{j_1, \dots, j_{m-1}\}$ is a sequence of $m - 1$ terms and $x_J = x_{j_1} \cdots x_{j_{m-1}}$. Then

$$\begin{aligned} U(\phi_{(1)})(\xi) - \xi &= \sum_{i,J} a_{i,J}\{(x_i + x'_i) U(\phi_{(1)})(x_J) - x_i x_J\} \\ &= \sum_i x_i \sum_J a_{iJ}(U(\phi_{(1)})(x_J) - x_J) + \sum_i x'_i \sum_J a_{iJ} U(\phi_{(1)})(x_J). \end{aligned}$$

Hence, for each i, $\sum_J a_{i,J}(U(\phi_{(1)})(x_J) - x_J) \in \operatorname{Int} T(x_1, \dots, x_r, x'_1, \dots, x'_r)$. Now the latter can be rewritten as $U(\phi_{(1)})(\sum_J a_{iJ} x_J) - \sum_J a_{iJ} x_J$, so the inductive hypothesis implies that each $a_{i,J} \in \mathbb{Z}$. This completes the induction and proves the preliminary result.

To prove the lemma, suppose that $(U(\phi_{(1)})(\xi) - \xi - U(\delta)(\xi)) \in \operatorname{Int} T(x_1, \dots, x_r, x'_1, \dots, x'_r)$. Write $\xi = \sum_{i,J} a_{iJ} x_i x_J$, so that

$$\begin{aligned} &U(\phi_{(1)})(\xi) - \xi - U(\delta)(\xi) \\ &\quad = \sum_i x_i \sum_J a_{iJ}(U(\phi_{(1)})(x_J) - x_J) + \sum_i x'_i \sum_J a_{iJ}(U(\phi_{(1)})(x_J) - U(\delta)(x_J)). \end{aligned}$$

Hence for each i, $\sum_J a_{iJ}(U(\phi_{(1)})(x_J) - x_J) \in \operatorname{Int} T(x_1, \dots, x'_1, \dots, x'_r)$. Again, this latter can be re-written $U(\phi_{(1)})(\sum_J a_{iJ} x_J) - \sum_J a_{iJ} x_J$, and the lemma follows from the preliminary result. $\square$

3.10. Proposition. *Let $\phi, \psi \in \operatorname{Int} \mathscr{C}(T(x_1, \dots, x_r))$ and let $\theta \in \operatorname{Aut}_* T(x_1, \dots, x_r)$. If $\theta * \phi = \psi$, then $\theta \in \operatorname{Int} \operatorname{Aut}_* T(x_1, \dots, x_r)$.*

Proof. We use induction over the number of generators x_i, with the inductive step proved by a secondary induction over the length. The main inductive hypothesis is that $\theta(x_i) \in \operatorname{Int} T(x_1, \dots, x_r)$ for all $i < k$. For $i = 1$ this is clearly true, since $\theta(x_1) = x_1$.

Now suppose that $\theta(x_i) \in \operatorname{Int} T(x_1, \ldots, x_r)$ for all $i < k$. Notice that this implies $\theta^{-1}(x_i)$ is also in $\operatorname{Int} T(x_1, \ldots, x_r)$ for all $i < k$. Write $\theta(x_k) = x_k + Q_2(x_k) + \cdots + Q_m(x_k)$, where $Q_s(x_k)$ is of length s. We will show that $Q_s(x_k) \in \operatorname{Int} T(x_1, \ldots, x_r)$ for each s, by induction over s. Our secondary induction hypothesis is that $Q_s(x_k) \in \operatorname{Int} T(x_1, \ldots, x_r)$ for each $s \le l$. This induction starts with $l = 1$ where the result is obvious. Now

$$\begin{aligned}\psi(x_k) &= (\theta * \phi)(x_k) = (\theta \sqcup \theta)\phi\theta^{-1}(x_k)\\ &= (\theta \sqcup \theta)\phi(x_k - \theta^{-1}Q_2(x_k) - \cdots - \theta^{-1}Q_m(x_k)).\end{aligned}$$

Working up to congruence modulo terms of length $\ge l + 2$, and letting P denote the perturbation of ϕ in a similar way to (1.1), we have

$$\begin{aligned}\psi(x_k) &\equiv (\theta \sqcup \theta)\phi(x_k - \theta^{-1}Q_2(x_k) - \cdots - \theta^{-1}Q_l(x_k) - Q_{l+1}(x_k))\\ &\equiv (\theta \sqcup \theta)\left(x_k + x_k' + \sum_{s=2}^{l+1} P_s(x_k) - \phi\theta^{-1}\sum_{s=2}^{l} Q_s(x_k) - U(\phi_{(1)})(Q_{l+1}(x_k))\right)\\ &\equiv x_k + \sum_{s=2}^{l} Q_s(x_k) + x_k' + U(\delta)\left(\sum_{s=2}^{l} Q_s(x_k)\right)\\ &\quad + (\theta \sqcup \theta)\left(\sum_{s=2}^{l+1} P_s(x_k) - \phi\theta^{-1}\sum_{s=2}^{l} Q_s(x_k)\right)\\ &\quad - (U(\phi_{(1)})(Q_{l+1}(x_k)) - Q_{l+1}(x_k) - U(\delta)(Q_{l+1}(x_k))).\end{aligned}$$

Now ψ and ϕ are in $\operatorname{Int}\mathscr{C}(T(x_1, \ldots, x_r))$, and $Q_s(x_k) \in \operatorname{Int} T(x_1, \ldots, x_r)$ for $1 \le s \le l$. Combining these facts with the remark above about θ^{-1}, we see that $U(\phi_{(1)})(Q_{l+1}(x_k)) - Q_{l+1}(x_k)) - U(\delta)(Q_{l+1}(x_k)) \in \operatorname{Int} T(x_1, \ldots, x_r, x_1', \ldots, x_r')$. Lemma 3.9 then gives that $Q_{l+1}(x_k) \in \operatorname{Int} T(x_1, \ldots, x_r)$, which completes the secondary induction. Hence $Q_s(x_k) \in \operatorname{Int} T(x_1, \ldots, x_r)$ for all s, and the main inductive step is proved. This completes the induction and the result is proved. □

For Lie algebra comultiplications, we do not prove as sharp a result as the above – see Remarks 3.13. However the following proposition is sufficient for our applications.

3.11. Proposition. *Let* ϕ, $\psi \in \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$ *and let* $\theta \in \operatorname{Aut}_* \mathbb{L}(x_1, \ldots, x_r)$. *If* $\theta * \phi = \psi$, *then* $sQ_s(x_j) \in \operatorname{Int}\mathbb{L}(x_1, \ldots, x_r)$, *for all* s and all j, *where* θ *is written* $\theta(x_j) = x_j + Q_2(x_j) + \cdots + Q_m(x_j)$ *as in* (1.2).

Proof. Let $i: \mathbb{L}(x_1, \ldots, x_r) \to T(x_1, \ldots, x_r)$ be the inclusion into the universal enveloping algebra. Now $U(\theta) \in \operatorname{Aut}_* T(x_1, \ldots, x_r)$ and $U(\phi), U(\psi) \in \operatorname{Int}\mathscr{C}(T(x_1, \ldots, x_r))$. Since $\theta * \phi = \psi$ implies $U(\theta) * U(\phi) = U(\psi)$, we apply Proposition 3.10 and conclude that $U(\theta) \in \operatorname{Int}\operatorname{Aut}_* T(x_1, \ldots, x_r)$. Then $i(Q_s(x_j)) \in \operatorname{Int} T(x_1, \ldots, x_r)$ for each s. According to [19, p. 281] (see also [14, p. 169]), the linear map ρ: $T(V) \to \mathbb{L}(V)$ defined by

$\rho(x_{j_1} \cdots x_{j_s}) = 1/s \quad [x_{j_1}, [\ldots, [x_{j_{s-1}}, x_{j_s}] \ldots]]$, is a left inverse for i. Hence $Q_s(x_j) = \rho i(Q_s(x_j))$, and since $i(Q_s(x_j)) \in \operatorname{Int} T(x_1, \ldots, x_r)$ we have for each generator x_j and each s, $sQ_s(x_j) \in \operatorname{Int} \mathbb{L}(x_1, \ldots, x_r)$. $\square$

Finally, we apply the preceding results to the map of orbit sets mentioned above.

3.12. Proposition. *The map of orbit sets*

$$q: \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))//\operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \ldots, x_r) \to \mathscr{C}(L)//\operatorname{Aut}_* L$$

induced by inclusion is finite-to-one, where $L = \mathbb{L}(x_1, \ldots, x_r)$.

Proof. We will write $[\phi]$ for the equivalence class of a comultiplication in the orbit set $\operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))//\operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \ldots, x_r)$ and $\{\phi\}$ for an equivalence class in the orbit set $\mathscr{C}(L)//\operatorname{Aut}_* L$. Suppose that $\{\phi\}$ is in the image of q with $\phi \in \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$. Let $\{\theta_1, \ldots, \theta_m\}$ be the subset of the set of automorphisms $\mathscr{F}$ described in Definition 3.5, consisting of those $\theta_j \in \mathscr{F}$ with $\theta_j * \phi \in \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$. We will show that $q^{-1}\{\phi\} = \{[\theta_1 * \phi], \ldots, [\theta_m * \phi]\}$. For suppose that $\psi \in \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$ and that $\{\phi\} = \{\psi\}$. Then there exists $\theta \in \operatorname{Aut}_* L$ with $\theta * \phi = \psi$. By Proposition 3.11 and Lemma 3.7, there exists $\tilde{\theta} \in \operatorname{Int}\operatorname{Aut}_* \mathbb{L}(x_1, \ldots, x_r)$ such that $\tilde{\theta} \circ \theta \in \mathscr{F}$. Now $\tilde{\theta} * \psi \in \operatorname{Int}\mathscr{C}(\mathbb{L}(x_1, \ldots, x_r))$, and $\tilde{\theta} * \psi = (\tilde{\theta} \circ \theta) * \phi$, so $\tilde{\theta} \circ \theta = \theta_j$ for some j. Hence $[\psi] = [\tilde{\theta} * \psi] = [\theta_j * \phi]$ for some j. $\square$

3.13. Remark. Observe that Proposition 3.10 asserts that the natural map

$$\operatorname{Int}\mathscr{C}(T(x_1, \ldots, x_r))//\operatorname{Int}\operatorname{Aut}_* T(x_1, \ldots, x_r) \to \mathscr{C}(T)//\operatorname{Aut}_* T$$

is one-to-one, where $T = T(x_1, \ldots, x_r)$. Thus Proposition 3.12 is a Lie algebra analogue of Proposition 3.10, but with a weakened conclusion. In fact, a stronger formulation of Proposition 3.12 is true. This uses the following Lie algebra counterpart to Lemma 3.9: If $\xi \in \mathbb{L}(V)$ is of length at least 3 and if $\phi_{(1)}(\xi) - \xi - \delta(\xi) \in \operatorname{Int} \mathbb{L}(V)$, then $\xi \in \operatorname{Int} L(V)$. The latter result can be proved with a rather delicate Hall basis argument. From this, one argues as in the proof of Proposition 3.10 and shows that in certain circumstances the map of orbit sets q is actually injective. For example, if each generator x_i has odd degree, then q is injective. However, Proposition 3.12 suffices for our purposes.

Combining Propositions 3.4 and 3.12, we have the following theorem.

3.14. Theorem. *Let X be a finite cogroup and let $L_X = \pi_\#(\Omega X) \otimes \mathbb{Q}$ be the rational homotopy Lie algebra. Then the map of orbit sets*

$$r: \mathscr{C}(X)//\mathscr{E}_*(X) \to \mathscr{C}(L_X)//\operatorname{Aut}_* L_X$$

induced by $g:\mathscr{C}(X) \to \mathscr{C}(L_X)$ *and* $h:\mathscr{E}_*(X) \to \mathrm{Aut}_* L_X$ *is finite-to-one. Consequently, if* $\mathscr{C}(L_X)//\mathrm{Aut}_* L_X$ *is finite, then* $\tilde{\mathscr{C}}(X)$ *is finite.*

3.15. Remark. The maps $g:\mathscr{C}(X) \to \mathscr{C}(L_X)$ and $h:\mathscr{E}(X) \to \mathrm{Aut}_* L_X$ also induce a map of equivalence classes $\tilde{r}$: $\tilde{\mathscr{C}}(X) \to \tilde{\mathscr{C}}(L_X)$. However, this map need not be finite-to-one, even when restricted to $\tilde{\mathscr{C}}_a(X)$. For instance, if $X = S^3 \vee S^5$, then $\tilde{\mathscr{C}}_a(X)$ has infinitely many elements (see Example 6.6(1)). On the other hand, an easy computation shows that $\tilde{\mathscr{C}}_a(L_X)$ has exactly two elements (cf. Example 5.4(1)). Since we prefer to compute in the setting of the rational homotopy Lie algebra whenever possible, this explains why we focus on the the orbit sets $\mathscr{C}(X)//\mathscr{E}_*(X)$ and $\mathscr{C}(L_X)//\mathrm{Aut}_* L_X$ rather than on the sets of equivalence classes $\tilde{\mathscr{C}}(X)$ and $\tilde{\mathscr{C}}(L_X)$.

In the following example we illustrate how, in applying the map r of Theorem 3.14, the loss of torsion may lead to strictly fewer equivalence classes of comultiplications. In what follows, and also in the examples of Section 6, we adopt the following notation: If X is the wedge of spheres $X = S^{n_1} \vee \cdots \vee S^{n_r}$, then $\iota_j \in \pi_{n_j}(X)$ denotes inclusion into the jth summand. Similarly we use ι_j and ι'_j to denote those elements of $\pi_{n_j}(X \vee X)$ given by inclusion into appropriate summands. The bracket in this context denotes *Whitehead product*. We will frequently use the fact that, if $\theta \in \pi_m(S^n)$ is a suspension, then left-additivity holds, i.e., for $f, g \in \pi_n(X)$ we have $(f+g)\circ\theta = f\circ\theta + g\circ\theta$.

3.16. Example. Let $X = S^7 \vee S^{19}$. Since X is of type (6, 18), one sees easily that $\mathscr{C}(L_X)$ is infinite. From [4, Theorem 3.14], however, it follows that $\mathscr{C}_a(L_X)$ and hence $\mathscr{C}_a(L_X)//\mathrm{Aut}_* L_X$ contains a single element. Since the map r: $\mathscr{C}(X)//\mathscr{E}_*(X) \to \mathscr{C}(L_X)//\mathrm{Aut}_* L_X$ restricts to a map r': $\mathscr{C}_a(X)//\mathscr{E}_*(X) \to \mathscr{C}_a(L_X)//\mathrm{Aut}_* L_X$, Theorem 3.14 now implies that $\mathscr{C}_a(X)//\mathscr{E}_*(X)$ is a finite set (cf. also Theorem 6.1 below). Indeed, the latter set contains exactly two elements, as we now show: According to [22, p. 187], we have $\pi_{19}(S^7) = 0$. It follows that $\mathscr{E}(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathscr{E}_*(X)$ consists of the identity. On the other hand, it follows from Hilton's theorem [11, Theorem A] that a typical element of $\mathscr{C}(X)$ can be written $\phi_{(a,b,c)}$, with $\phi_{(a,b,c)}(\iota_1) = \iota_1 + \iota'_1$ and

$$\phi_{(a,b,c)}(\iota_2) = \iota_2 + \iota'_2 + a[\iota_1, \iota'_1]\circ\theta + b[\iota_1,[\iota_1,\iota'_1]] + c[\iota'_1,[\iota_1,\iota'_1]],$$

for integers a, b, c with $\theta \in \pi_{19}(S^{13}) \cong \mathbb{Z}_2$ denoting the generator [22, p. 186]. Since θ is a suspension, a may be reduced modulo 2. Now if $\phi_{(a,b,c)} \in \mathscr{C}_a(X)$, then $b = c = 0$. This follows from [4, Theorem 6.5.(ii)] and the well-known fact that, for η an odd-dimensional homotopy element, the triple Whitehead product $[\eta,[\eta,\eta]]$ is zero. Hence there are two elements in $\mathscr{C}_a(X)$, represented by $\phi_{(0,0,0)}$ and $\phi_{(1,0,0)}$. By the previous remarks $\mathscr{E}_*(X)$ is trivial, and so $\mathscr{C}_a(X)//\mathscr{E}_*(X)$ contains two equivalence classes.

We remark that the calculation can be continued to show that $\mathscr{E}(X)$ acts trivially on $\mathscr{C}_a(X)$, and so $\tilde{\mathscr{C}}_a(S^7 \vee S^{19})$ also has exactly two elements.

4. Quadratic Lie algebra comultiplications

The purpose of this section is to prove a result (Theorem 4.4) that essentially shows an associative comultiplication is determined up to isomorphism by its quadratic part. This result, which is of interest in its own right, will be used in Section 5.

If ϕ is a comultiplication of $L = \mathbb{L}(x_1, \ldots, x_r)$ we write, as in (1.1), $\phi(x_i) = x_i + x_i' + \sum_{s \geq 2} P_s(x_i)$ where $P_s(x_i)$ is the perturbation term of length s.

4.1. Definition. The *t-fold part* of ϕ is the comultiplication $\phi_{(t)}$ defined by

$$\phi_{(t)}(x_i) = x_i + x_i' + \sum_{s=2}^{t} P_s(x_i).$$

Note that this depends on the choice of a basis $x_1, \ldots, x_r$ of L. The 2-fold part $\phi_{(2)}$ of ϕ is called the *quadratic part* of ϕ, and a comultiplication ϕ is called *quadratic* if $\phi = \phi_{(2)}$. This is consistent with the definition of $\phi_{(1)}$ in Notation 3.8.

As before let $L \sqcup L \sqcup L = \mathbb{L}(x_1, \ldots, x_r, x_1', \ldots, x_r', x_1'', \ldots, x_r'')$. The following notation will be used extensively in this section.

4.2. Notation. Define homomorphisms $\beta, \gamma, \delta: L \sqcup L \to L \sqcup L \sqcup L$ by $\beta(x_i) = x_i + x_i'$, $\beta(x_i') = x_i''$; $\gamma(x_i) = x_i$, $\gamma(x_i') = x_i' + x_i''$; $\delta(x_i) = x_i'$, $\delta(x_i') = x_i''$. As in Notation 3.8 we also regard δ: $L \to L \sqcup L$. Note that $\beta = \phi_{(1)} \sqcup 1$ and $\gamma = 1 \sqcup \phi_{(1)}$.

4.3. Proposition. *Let $\chi \in \mathbb{L}(x_1, \ldots, x_r, x_1', \ldots, x_r')$ be of length $s \geq 3$. If $\chi + \beta(\chi) = \gamma(\chi) + \delta(\chi)$, then $\chi = \phi_{(1)}(\xi) - \xi - \delta(\xi)$ for some $\xi \in \mathbb{L}(x_1, \ldots, x_r)$ of length s.*

Proof. This is an immediate consequence of the proof of Theorem 3.11 in [4]. One replaces $P_r(x_j)$ in [4, 3.12] by χ, and argues as in the rest of that proof. $\square$

We now state and prove the main result of this section.

4.4. Theorem. *If ϕ and ψ are associative comultiplications of the Lie algebra $L = \mathbb{L}(x_1, \ldots, x_r)$ such that $\phi_{(2)} = \psi_{(2)}$, then there exists $\theta \in \mathrm{Aut}_* L$ such that $\theta * \psi = \phi$.*

Proof. We construct θ inductively over the generators x_i. We also use a secondary inductive argument. The primary inductive hypothesis is the following: There exists $\theta^{(k-1)} \in \mathrm{Aut}_* L$ such that $(\theta^{(k-1)} * \psi)_{(2)} = \psi_{(2)} = \phi_{(2)}$ and $\theta^{(k-1)} * \psi$ and ϕ agree on all generators x_i with $i < k$. The induction begins by taking $\theta^{(1)} = 1$, the identity automorphism.

Now suppose $\theta^{(k-1)}$ exists. We construct $\theta^{(k)}$ by a secondary induction over the bracket length. We write ϕ as in (1.1) and assume inductively that for some $m \geq 2$, there exists $\eta_{(m)} \in \mathrm{Aut}_* L$ such that

(a) $\eta_{(m)}$ has the form

$$\eta_{(m)}(x_i) = \begin{cases} x_i & \text{if } i \neq k, \\ x_k - \sum_{s=3}^{m} \xi_s & \text{if } i = k, \end{cases}$$

where each ξ_s is of length s and

(b) if R is the perturbation of the comultiplication $\eta_{(m)} * \theta^{(k-1)} * \psi$, namely,

$$(\eta_{(m)} * \theta^{(k-1)} * \psi)(x_k) = x_k + x_k' + \sum_{s \geq 2} R_s(x_k), \tag{4.7}$$

then $P_s(x_k) = R_s(x_k)$ for $s = 2, \dots, m$, where P is the perturbation of ϕ.

Claim 1. *$P_{m+1}(x_k) = R_{m+1}(x_k) + \phi_{(1)}(\xi_{m+1}) - \xi_{m+1} - \delta(\xi_{m+1})$ for some $\xi_{m+1} \in \mathbb{L}(x_1, \dots, x_r)$ of length $m+1$.*

Proof of Claim 1. Since ϕ is associative, $(\phi \sqcup 1)\phi = (1 \sqcup \phi)\phi$, we have

$$\sum_{s \geq 2} P_s(x_k) + (\phi \sqcup 1)\Big(\sum_{s \geq 2} P_s(x_k)\Big) = (1 \sqcup \phi)\Big(\sum_{s \geq 2} P_s(x_k)\Big) + \delta\Big(\sum_{s \geq 2} P_s(x_k)\Big). \tag{4.8}$$

Similarly, since ψ, and hence $\eta_{(m)} * \theta^{(k-1)} * \psi$, is associative, (4.7) yields

$$\sum_{s \geq 2} R_s(x_k) + (\phi \sqcup 1)\Big(\sum_{s \geq 2} R_s(x_k)\Big) = (1 \sqcup \phi)\Big(\sum_{s \geq 2} R_s(x_k)\Big) + \delta\Big(\sum_{s \geq 2} R_s(x_k)\Big). \tag{4.9}$$

We subtract (4.9) from (4.8) and obtain

$$\begin{aligned} &\sum_{s \geq m+1} (P_s(x_k) - R_s(x_k)) + (\phi \sqcup 1)\Big(\sum_{s \geq m+1} (P_s(x_k) - R_s(x_k))\Big) \\ &= (1 \sqcup \phi)\Big(\sum_{s \geq m+1} (P_s(x_k) - R_s(x_k))\Big) + \delta\Big(\sum_{s \geq m+1} (P_s(x_k) - R_s(x_k))\Big). \end{aligned} \tag{4.10}$$

We extract the length $m+1$ terms from (4.10) and get

$$\begin{aligned} &P_{m+1}(x_k) - R_{m+1}(x_k) + \beta(P_{m+1}(x_k) - R_{m+1}(x_k)) \\ &= \gamma(P_{m+1}(x_k) - R_{m+1}(x_k)) + \delta((P_{m+1}(x_k) - R_{m+1}(x_k)) \end{aligned}$$

Claim 1 now follows from Proposition 4.3. □

Proof of Theorem 4.4 (*continued*). With ξ_{m+1} as in Claim 1, define $\eta_{m+1} \in \mathrm{Aut}_* L$ by

$$\eta_{m+1}(x_i) = \begin{cases} x_i & \text{if } i \neq k, \\ x_k - \xi_{m+1} & \text{if } i = k. \end{cases}$$

We let T be the perturbation of the comultiplication $\eta_{m+1} * (\eta_{(m)} * \theta^{(k-1)} * \psi)$, so

$$(\eta_{m+1} * (\eta_{(m)} * \theta^{(k-1)} * \psi))(x_k) = x_k + x_k' + \sum_{s \geq 2} T_s(x_k).$$

Claim 2. *For* $s = 2, \dots, m+1$, $T_s(x_k) = P_s(x_k)$.

Proof of Claim 2. We have

$$(\eta_{m+1} * \eta_{(m)} * \theta^{(k-1)} * \psi)(x_k) = (\eta_{m+1} \sqcup \eta_{m+1})((\eta_{(m)} * \theta^{(k-1)} * \psi)(x_k) + \phi(\xi_{m+1})), \tag{4.11}$$

since for generators x_j with $j < k$, $(\eta_{(m)} * \phi^{(k-1)} * \psi)(x_j) = (\theta^{(k)} * \psi)(x_j) = \phi(x_j)$. Take all the terms of length $\le m+1$ from (4.11) and get

$$\begin{aligned} x_k + x_k' + \sum_{s=2}^{m+1} T_s(x_k) &= (\eta_{m+1} \sqcup \eta_{m+1})\left(x_k + x_k' + \sum_{s=2}^{m} P_s(x_k) + R_{m+1}(x_k) + \phi_{(1)}(\xi_{m+1})\right) \\ &= x_k - \xi_{m+1} + x_k' - \delta(\xi_{m+1}) + \sum_{s=2}^{m} P_s(x_k) + R_{m+1}(x_k) + \phi_{(1)}(\xi_{m+1}). \end{aligned}$$

Thus,

$$\sum_{s=2}^{m+1} T_s(x_k) = \sum_{s=2}^{m} P_s(x_k) + R_{m+1}(x_k) + \phi_{(1)}(\xi_{m+1}) - \xi_{m+1} - \delta(\xi_{m+1}) = \sum_{s=2}^{m+1} P_s(x_k).$$

This proves Claim 2. $\square$

Proof of Theorem 4.4 (*continued*). By construction, $\eta_{m+1} \circ \eta_{(m)}$ has the form

$$(\eta_{m+1} \circ \eta_{(m)})(x_i) = \begin{cases} x_i & \text{if } i \neq k, \\ x_k - \sum_{s=3}^{m+1} \xi_s & \text{if } i = k. \end{cases}$$

We set $\eta_{(m+1)} = \eta_{m+1} \circ \eta_{(m)}$ to complete the inductive step of the secondary induction. We start the secondary induction by setting $\eta_{(2)}$ equal to the identity homomorphism and note that $\eta_{(2)}$ satisfies the desired condition since $(\theta^{(k)} * \psi)_{(2)} = \phi_{(2)}$. Therefore there exists $\eta \in \mathrm{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ such that

$$\eta(x_i) = \begin{cases} x_i & \text{if } i \neq k, \\ x_k - \sum_{s \ge 3} \xi_s & \text{if } i = k, \end{cases} \tag{4.12}$$

where ξ_s is of length s and $\eta * \theta^{(k-1)} * \psi$ and ϕ agree on all generators x_i with $i \le k$. It follows from (4.12) that $(\eta * \theta^{(k-1)} * \psi)_{(2)} = (\theta^{(k-1)} * \psi)_{(2)}$ and thus equals $\phi_{(2)}$ by the primary inductive hypothesis. By setting $\theta^{(k)} = \eta \circ \theta^{(k-1)}$, we complete the inductive step. Finally, the theorem is proved by putting $\theta = \theta^{(r)}$, since r is the number of generators of $L = \mathbb{L}(x_1, \dots, x_r)$. $\square$

4.5. Remark. There is an analogue of Theorem 4.4 for tensor algebras as follows: If ϕ and ψ are associative comultiplications of the tensor algebra $T = T(x_1, \dots, x_r)$ such that $\phi_{(2)} = \psi_{(2)}$, then there exists $\theta \in \mathrm{Aut}_* T$ such that $\theta * \psi = \phi$. The proof is obtained by carrying over the proof of Theorem 4.4 *mutatis mutandis* to tensor algebras. We apply this tensor algebra result in [5].

We finish this section with an example that illustrates Theorem 4.4 cannot be improved. If an associative comultiplication $\phi \in \mathscr{C}_a(L)$ has quadratic part $\phi_{(2)}$ which itself is an associative comultiplication, then Theorem 4.4 implies that ϕ is equivalent to its own quadratic part $\phi_{(2)}$. Our example shows that in general this situation need not pertain.

Let L be the Lie algebra $\mathbb{L}(x_p, y_{2p}, z_{3p})$ where subscripts denote degrees, and assume that $|x_p| = p$ is even. Let ϕ be any comultiplication of L with perturbation P and write

$$P(x) = 0, \qquad P(y) = a[x, x']$$
$$\text{and} \quad P(z) = b[x, y'] + c[x', y] + d[x, [x, x']] + e[x', [x, x']]$$

for some $a,b,c,d,e \in \mathbb{Q}$. If $\theta \in \operatorname{Aut} L$, then

$$\theta(x) = \alpha x, \qquad \theta(y) = \beta y \quad \text{and} \quad \theta(z) = \gamma z + \delta[x, y]$$

for α, β, γ, $\delta \in \mathbb{Q}$ with α, β, γ non-zero.

4.6. Lemma. *With L, ϕ and θ as above:*

(1) *ϕ is associative $\Leftrightarrow ab + e = 2d$ and $ab + ac = d + e$.*

(2) (a) $(\theta * \phi)(x) = x + x'$,

(b) $(\theta * \phi)(y) = y + y' + \dfrac{a\alpha^2}{\beta}[x, x']$,

(c) $$(\theta * \phi)(z) = z + z' + \frac{b\alpha\beta - \delta}{\gamma}[x, y'] + \frac{c\alpha\beta - \delta}{\gamma}[x', y] + \frac{d\alpha^3\beta - a\alpha^2\delta}{\beta\gamma}[x, [x, x']] + \frac{e\alpha^3\beta - a\alpha^2\delta}{\beta\gamma}[x', [x, x']].$$

Proof. The proof is a straightforward but long calculation, and hence omitted. □

4.7. Example. Define a comultiplication ϕ by taking $a = 1$, $b = 2$, $c = -1$, $d = 1$, $e = 0$ in the above. Then ϕ is associative, by Lemma 4.6, but it cannot be equivalent to any quadratic comultiplication. To see this, suppose that $\theta * \phi$ is quadratic, where $\theta \in \operatorname{Aut} L$ is as described above. Then Lemma 4.6 implies $\alpha^3\beta - \alpha^2\delta = 0$ and $\alpha^2\delta = 0$. From the latter, $\delta = 0$. This implies $\alpha^3\beta = 0$ which is impossible. Notice, in particular, that ϕ cannot be equivalent to its own quadratic part. Indeed, it follows from Lemma 4.6 that $\phi_{(2)}$ is not associative.

5. Lie algebra comultiplications

In this section we consider comultiplications of Lie algebras and prove that certain orbit sets are finite (Proposition 5.3), by using Theorem 4.4 on the quadratic part of comultiplications. This result together with Theorem 3.14 will yield our main theorem in Section 6 (Theorem 6.1) on the orbit set of comultiplications of a finite cogroup.

We adopt the following standard convention for taking a sum over an index set: Whenever the index set is empty, the sum is defined to be zero.

5.1. Proposition. *Let ϕ be a comultiplication in $\mathscr{C}_a(\mathbb{L}(x, \dots, x_r))$ with perturbation P such that*

$$P_2(x_i) = \sum_{j, n_j \text{odd}} a^i_j[x_j, x'_j] + \sum_{j<k} a^i_{j,k}([x_j, x'_k] + [x'_j, x_k])$$

for $a^i_j, a^i_{j,k} \in \mathbb{Q}$. Then ϕ is equivalent, by an element in $\operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$, to the comultiplication $\phi_{(1)}$.

Proof. Define $\theta \in \operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ by

$$\theta(x_i) = x_i + \sum_{j, n_j \text{odd}} \frac{a^i_j}{2}[x_j, x_j] - \sum_{j<k} a^i_{j,k}[x_j, x_k].$$

We determine $(\theta * \phi)_{(2)}$. Observe that modulo terms of length ≥ 3,

$$\theta^{-1}(x_i) \equiv x_i - \sum_{j, n_j \text{odd}} \frac{a^i_j}{2}[x_j, x_j] - \sum_{j<k} a^i_{j,k}[x_j, x_k].$$

Therefore, working up to congruence modulo terms of length ≥ 3, we have

$$\begin{aligned}(\theta * \phi)(x_i) \equiv (\theta \sqcup \theta)\Bigg(& x_i + x'_i + \sum_{j, n_j \text{odd}} a^i_j[x_j, x'_j] + \sum_{j<k} a^i_{j,k}([x_j, x'_k] + [x'_j, x_k]) \\ & - \sum_{j, n_j \text{odd}} \frac{a^i_j}{2}[x_j + x'_j, x_j + x'_j] - \sum_{j<k} a^i_{j,k}[x_j + x'_j, x_k + x'_k]\Bigg).\end{aligned}$$

Hence $(\theta * \phi)_{(2)}(x_i) = x_i + x'_i$. Since $\theta * \phi$ and $\phi_{(1)}$ are associative, it follows from Theorem 4.4 that there exists $\theta' \in \operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ such that $\theta' * (\theta * \phi) = \phi_{(1)}$. Thus ϕ is equivalent to $\phi_{(1)}$ via $\theta'\theta \in \operatorname{Aut}_* \mathbb{L}(x_1, \dots, x_r)$. $\square$

Recall that $\mathscr{C}_{ac}(L) \subseteq \mathscr{C}_a(L)$ consists of the Lie algebra comultiplications of L which are associative and commutative.

5.2. Corollary. *Let $L = \mathbb{L}(x_1, \dots, x_r)$ with $|x_i| = n_i$. Suppose that, for each i,* (a) *$n_i \neq n_j + n_j$ for every j, k with $j < k$ and* (b) *$n_i \neq 2n_j$ for every j with n_j even. Then the map of orbit sets λ: $\mathscr{C}_{ac}(L)//\operatorname{Aut}_* L \to \mathscr{C}_a(L)//\operatorname{Aut}_* L$ induced by inclusion is a bijection.*

Proof. Clearly λ is one-one. Now let ϕ be any associative comultiplication of L. Under our assumptions, ϕ satisfies the hypotheses of Proposition 5.1 with the $a^i_{j,k}$ all zero. Thus ϕ is equivalent to the associative and commutative comultiplication $\phi_{(1)}$. $\square$

5.3. Proposition. *Let $L = \mathbb{L}(x_1, \dots, x_r)$ with $|x_i| = n_i$.*

(1) *If for each i,* (a) $n_i \neq n_j + n_k$ *for every j, k with $j < k$ and* (b) $n_i \neq 2n_j$ *for every j with n_j even, then the orbit set $\mathscr{C}_a(L)//\mathrm{Aut}_* L$ contains a single element.*

(2) *The orbit set $\mathscr{C}_{ac}(L)//\mathrm{Aut}_* L$ contains a single element.*

Proof. By Corollary 5.2, it suffices to show (2). Let $\phi \in \mathscr{C}_{ac}(\mathbb{L}(x_1, \dots, x_r))$ have pertubation P. Since ϕ is commutative, $TP_2(x_i) = P_2(x_i)$, where T is the switching homomorphism. From this it follows that ϕ satisfies the hypotheses of Proposition 5.1. Thus ϕ is equivalent by an element of $\mathrm{Aut}_* \mathbb{L}(x_1, \dots, x_r)$ to the comultiplication $\phi_{(1)}$. □

5.4. Remarks. (1) The conclusion of Proposition 5.3(1) need not be true without the hypothesis on n_i. For example, if we set $L = \mathbb{L}(x_n, y_{2n})$, where subscripts denote degree and n is even, then $\mathrm{Aut}_* L$ consists of the identity automorphism. But for every m, the comultiplication ϕ defined by $\phi(y) = y + y' + m[x, x']$ is in $\mathscr{C}_a(L)$. Thus the orbit set $\mathscr{C}_a(L)//\mathrm{Aut}_* L$ is infinite.

(2) Under hypothesis (1) of Proposition 5.3, every associative comultiplication on L is equivalent to $\phi_{(1)}$. Thus if ϕ is an associative comultiplication of a Lie algebra L which satisfies (1), then L is 'primitively generated' with respect to ϕ.

The following is an immediate consequence of Proposition 5.3.

5.5. Corollary (cf. [4, Theorem 3.18] and [9, p. 8]). *If $L = \mathbb{L}(x_1, \dots, x_r)$ with $|x_i|$ odd for every i, then $\mathscr{C}_a(L)//\mathrm{Aut}_* L$ consists of a single element. Thus any two associative comultiplications of L are equivalent via an automorphism in* $\mathrm{Aut}_* L$. □

5.6. Remarks. Corollary 5.5 and part (2) of Proposition 5.3 are duals of well-known results about diagonals on commutative graded algebras. Corollary 5.5 is dual to the Leray–Samelson theorem [17, 7.20], which states that a commutative Hopf algebra with generators in odd degrees is primitively generated. Proposition 5.3(2) is dual to the result of Milnor–Moore [17, 4.18] that a commutative Hopf algebra with a commutative diagonal is primitively generated.

6. Comultiplications of a finite cogroup

We recall from Section 2 that a finite co-H-space X has the rational homotopy type of a wedge of spheres, $X_{\mathbb{Q}} \equiv S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_r+1}$ for integers $n_1 \leq \cdots \leq n_r$, and that $(n_1, \dots, n_r)$ is the type of X.

The following is the main result of this section.

6.1. Theorem. *Let X be a finite cogroup of type $(n_1, \dots, n_r)$.*

(1) *If for each i,* (a) *$n_i \neq n_j + n_k$ for every j, k with $j < k$ and* (b) *$n_i \neq 2n_j$ for every j with n_j even, then the set of orbits $\mathscr{C}_{\mathrm{a}}(X)//\mathscr{E}_*(X)$ is finite. Consequently $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$ is finite.*

(2) *The set of orbits $\mathscr{C}_{\mathrm{ac}}(X)//\mathscr{E}_*(X)$ is finite. Consequently $\tilde{\mathscr{C}}_{\mathrm{ac}}(X)$ is finite.*

Proof. (1) Let $L_X = \pi_{\#}(\Omega X) \otimes \mathbb{Q} = \mathbb{L}(x_1, \dots, x_r)$ with $|x_i| = n_i$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathscr{C}_{\mathrm{a}}(X)//\mathscr{E}_*(X) & \xrightarrow{r'} & \mathscr{C}_{\mathrm{a}}(L_X)//\mathrm{Aut}_* L_X \\
\downarrow & & \downarrow \\
\mathscr{C}(X)//\mathscr{E}_*(X) & \xrightarrow{r} & \mathscr{C}(L_X)//\mathrm{Aut}_* L_X
\end{array}$$

where r is as in Theorem 3.14, the vertical maps are induced by inclusion and r' is the restriction of r. Since r is a finite-to-one map by Theorem 3.14, r' is a finite-to-one map. By Proposition 5.3, $\mathscr{C}_{\mathrm{a}}(L_X)//\mathrm{Aut}_* L_X$ consists of a single element. Thus $\mathscr{C}_{\mathrm{a}}(X)//\mathscr{E}_*(X)$ is finite. The second assertion of (1) follows from the first since the map $\mathscr{C}_{\mathrm{a}}(X)//\mathscr{E}_*(X) \to \mathscr{C}_{\mathrm{a}}(X)//\mathscr{E}(X) = \tilde{\mathscr{C}}_{\mathrm{a}}(X)$ induced by the inclusion $\mathscr{E}_*(X) \subseteq \mathscr{E}(X)$ is onto.

Part (2) is proved analogously to part (1). □

6.2. Corollary. *If X is a finite cogroup of type $(n_1, \dots, n_r)$ with all n_i odd, then $\tilde{\mathscr{C}}_{\mathrm{a}}(X)$ is finite.*

6.3. Remark. In [9, Theorem I], Curjel proves that there are finitely many equivalence classes of homotopy-associative multiplications of a finite H-space. We regard Corollary 6.2 as dual to Curjel's theorem, and justify this in the following way. A finite H-space has an oddly generated rational cohomology algebra. A finite co-H-space, on the other hand, has an oddly generated rational homotopy Lie algebra if and only if it has type $(n_1, \dots, n_r)$ with all n_i odd.

6.4. Remark. Clearly Theorem 6.1 holds with the set of homotopy classes of comultiplications of X whose rationalization is homotopy-associative replacing the set of homotopy-associative comultiplications of X. A similar remark holds for comultiplications which are rationally both homotopy-associative and homotopy-commutative.

We illustrate Theorem 6.1 with several examples. The strength of our results becomes apparent when the ad hoc approach of Example 3.16 is taken instead, as in the following example.

6.5. Example. Let $Y = S^3 \vee S^4 \vee S^8$. Theorem 6.1 implies $\tilde{\mathscr{C}}_{\mathrm{a}}(Y)$ is finite, since Y is of type (2, 3, 7). Alternatively, as in Example 3.16 use Hilton's theorem to write a typical element of $\mathscr{C}(Y)$ as ϕ_λ for $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12})$.

with $\phi_\lambda(\iota_1) = \iota_1 + \iota'_1$, $\phi_\lambda(\iota_2) = \iota_2 + \iota'_2$, and

$$\begin{aligned}\phi_\lambda(\iota_3) = \iota_3 + \iota'_3 &+ \lambda_1[\iota_1, \iota'_1] \circ \alpha + \lambda_2[\iota_1, \iota'_2] \circ \beta + \lambda_3[\iota'_1, \iota_2] \circ \beta + \lambda_4[\iota_2, \iota'_2] \circ \gamma \\ &+ \lambda_5[\iota_1, [\iota_1, \iota'_1]] \circ \gamma + \lambda_6[\iota'_1, [\iota_1, \iota'_1]] \circ \gamma \\ &+ \lambda_7[\iota_1, [\iota_1, \iota'_2]] + \lambda_8[\iota_1, [\iota'_1, \iota'_2]] + \lambda_9[\iota'_1, [\iota_1, \iota'_2]] \\ &+ \lambda_{10}[\iota'_1, [\iota_1, \iota_2]] + \lambda_{11}[\iota_1, [\iota'_1, \iota_2]] + \lambda_{12}[\iota'_1, [\iota'_1, \iota_2]].\end{aligned}$$

In this last equation α, β and γ denote generators, respectively, of $\pi_8(S^5) \cong \mathbb{Z}_{24}$, $\pi_8(S^6) \cong \mathbb{Z}_2$ and $\pi_8(S^7) \cong \mathbb{Z}_2$ [22, p. 186], λ_1 is reduced modulo 24 and $\lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 are reduced modulo 2. We use the fact that all the generators are suspensions to describe $\phi_\lambda(\iota_3)$ in this way. Now consider the restrictions on the λ_j in order that ϕ_λ be homotopy-associative: By a straightforward argument using commutativity and the Jacobi identity for the Whitehead product as in [4, Example 6.9], we see that $\lambda_5 = \lambda_6 = 0$. By applying [4, Theorem 6.5(ii)], in conjunction with the result mentioned in Remark 3.13, it is possible to show that $\lambda_j = \lambda_7$ for $j = 8, 9, 10, 11$ and 12. It now follows that we can write

$$\begin{aligned}\phi_\lambda(\iota_3) = \iota_3 + \iota'_3 &+ \lambda_1[\iota_1, \iota'_1] \circ \alpha + \lambda_2[\iota_1, \iota'_2] \circ \beta + \lambda_3[\iota'_1, \iota_2] \circ \beta + \lambda_4[\iota_2, \iota'_2] \circ \gamma \\ &+ \lambda_7\{[\iota_1 + \iota'_1, [\iota_1 + \iota'_1, \iota_2 + \iota'_2]] - [\iota_1, [\iota_1, \iota_2]] - [\iota'_1, [\iota'_1, \iota'_2]]\}.\end{aligned}$$

Furthermore, a straightforward computation shows that any comultiplication of this form is homotopy-associative, so $\mathscr{C}_a(Y)$ contains infinitely many elements. Proceeding as in Theorem 4.4, we now observe that if ϕ_λ is as in the last expressions, and if $\theta \in \mathscr{E}_*(Y)$ is given by $\theta(\iota_1) = \iota_1, \theta(\iota_2) = \iota_2$ and $\theta(\iota_3) = \iota_3 + \lambda_7[\iota_1, [\iota_1, \iota_2]]$, then $\theta * \phi_\lambda = \phi_{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, 0, 0, 0, 0, 0, 0, 0, 0)}$. Hence $\tilde{\mathscr{C}}_a(Y)$ is finite.

In Example 3.16 we had $\mathscr{C}_a(X)$ finite with $\mathscr{E}(X)$ acting trivially. In this example, however, $\mathscr{C}_a(Y)$ is infinite and it is necessary to take into account the action of $\mathscr{E}(Y)$. To do so, we basically mimic the proofs of results such as Theorem 4.4. This example suggests that any argument to show $\tilde{\mathscr{C}}_a(Y)$ is finite, which is to be generally applicable, will in one form or another use some of the key ingredients of Theorem 6.1.

From the previous working, if $Y = S^3 \vee S^4 \vee S^8$, then $\tilde{\mathscr{C}}_a(Y)$ contains at most 24×2^3 elements. It is natural to ask whether we can be more precise. Our calculations may be continued along similar lines to show that $\tilde{C}_a(Y)$ contains at most $24 \times 2^2 = 96$ elements. Standard methods such as Ganea's result [10, Corollary 3.5] do not apply here. It would be interesting to know the exact number of elements in $\tilde{\mathscr{C}}_a(S^3 \vee S^4 \vee S^8)$.

Our next examples show the conclusion of Theorem 6.1(1) may not hold, if the hypothesis is relaxed. We will see that even when X is a wedge of two or three spheres, there are many interesting possibilities, and our examples will be of this type.

First note that the hypothesis of Theorem 6.1(1) may fail to hold in one of two ways: either $n_i = 2n_j$ with n_j even or $n_i = n_j + n_k$ with $j < k$.

6.6. Examples. (1) Let $X = S^{n+1} \vee S^{2n+1}$, where n is even. We show that $\tilde{\mathscr{C}}_a(X)$ is infinite. By [4, Theorem 6.6], $\mathscr{C}_a(X)$ is infinite. On the other hand, $\mathscr{E}(X)$ is a finite group (see, for example, Proposition A.2 below). Therefore the orbit set $\tilde{\mathscr{C}}_a(X)$ is infinite.

(2) Let $X = S^{m+1} \vee S^{n+1} \vee S^{m+n+1}$ with $m < n$ and $n \neq 2m$. We show $\tilde{\mathscr{C}}_a(X)$ is infinite. Define for every pair of integers s, t a homotopy-associative comultiplication $\phi_{(s,t)}$ of X by $\phi_{(s,t)}(\iota_1) = \iota_1 + \iota'_1, \phi_{(s,t)}(\iota_2) = \iota_2 + \iota'_2$ and $\phi_{(s,t)}(\iota_3) = \iota_3 + \iota'_3 + s[\iota_1, \iota'_2] + t[\iota'_1, \iota_2]$. We claim $\phi_{(s,t)} \sim \phi_{(u,v)}$ implies $|s - t| = |u - v|$. Assuming this claim, we easily see that $\tilde{\mathscr{C}}_a(X)$ is infinite since, for example, the comultiplications $\phi_{(s,0)}$ represent distinct equivalence classes for each s. We now prove the claim: Suppose $f \in \mathscr{E}(X)$ and $f * \phi_{(s,t)} = \phi_{(u,v)}$. Then f has the form $f(\iota_1) = p\iota_1, f(\iota_2) = q\iota_2 + \alpha$ and $f(\iota_3) = r\iota_3 + k[\iota_1, \iota_2] + \beta$, where each of p, q and r is either 1 or -1, k is an integer and $\alpha \in \pi_{n+1}(X)$ and $\beta \in \pi_{m+n+1}(X)$ are torsion elements. Now a straightforward calculation, working up to congruence modulo torsion, yields $u - v = pqr^{-1}(s - t)$.

Finally we show that, although $\tilde{\mathscr{C}}_{ac}(X)$ is always finite, the set of equivalence classes of homotopy-commutative comultiplications $\tilde{\mathscr{C}}_c(X)$ could be infinite.

6.7. Example. Let $X = S^{n+1} \vee S^{rn+1}, r \geq 3$. Then by [4, p. 103 and Proposition 4.2], $\mathscr{C}_c(X)$ is infinite. However, $\mathscr{E}(X)$ is finite and thus $\tilde{\mathscr{C}}_c(X)$ is infinite.

Appendix. The group of self-homotopy equivalences that induce the identity on homology

Let Y be a 1-connected, finite complex. In this appendix we prove that $\mathscr{E}_*(Y)$ and all of its subgroups are finitely generated. An analogous result for homotopy groups appears in [2]. Our argument depends heavily on the work of Maruyama.

We use the notation of [16]. For $n \geq 3$, let $V = S_1^{n-1} \vee \cdots \vee S_r^{n-1}$ be a wedge of r $(n-1)$-spheres, A a 1-connected, finite complex of dimension $\leq n - 1$, $f\colon V \to A$ an arbitrary map and X the mapping cone of f. Then there is a cofibre sequence

$$V \xrightarrow{f} A \xrightarrow{i} X,$$

where i is the inclusion. Assume $V_1 = S_1^{n-1} \vee \cdots \vee S_m^{n-1} \subseteq V$ satisfies $\ker f_* = H_{n-1}(V/V_1) = H_{n-1}(S_{m+1}^{n-1} \vee \cdots \vee S_r^{n-1})$, where $f_*\colon H_{n-1}(V) \to H_{n-1}(A)$ is the induced homomorphism. Denote by $G(V)$ the subgroup of $\mathscr{E}(V)$ consisting of all elements of the form $1 + \gamma$ for $\gamma \in [V_1, V/V_1]$. Define a subgroup $\bar{\mathscr{E}}_*(A)$ of $\mathscr{E}_*(A)$ by

$$\bar{\mathscr{E}}_*(A) = \{g \in \mathscr{E}_*(A) \mid gf \simeq f\delta \text{ for some } \delta \in G(V)\}.$$

Consider the map $i_*\colon [A, A] \to [A, X]$ induced by i and the set $i_*\bar{\mathscr{E}}_*(A) \subseteq [A, X]$.

A.1. Lemma. *The set $i_*\bar{\mathscr{E}}_*(A)$ can be given group structure so that $i_*: \bar{\mathscr{E}}_*(A) \to i_*\bar{\mathscr{E}}_*(A)$ is an epimorphism.*

Proof. For $g, h \in \bar{\mathscr{E}}_*(A)$, define $i_*(g) \cdot i_*(h) = i_*(gh)$. We show that this is well-defined. Suppose $i_*(g) = i_*(g')$ and $i_*(h) = i_*(h')$. Then $igf = if\delta$ for some $\delta \in G(V)$. But $if\delta$ is the constant map and so ig induces a map $\tilde{g}$: $X \to X$ such that $\tilde{g}i = ig$. Thus $i_*(gh) = \tilde{g}ih = \tilde{g}ih' = igh' = ig'h' = i_*(g'h')$. This shows that the multiplication in $i_*\bar{\mathscr{E}}_*(A)$ is well-defined. The rest of Lemma A.1 now follows easily. □

Next consider the subgroup $i_*[\Sigma V, A]$ of $[\Sigma V, X]$.

A.2. Proposition. *There are homomorphisms λ: $i_*[\Sigma V, A] \to \mathscr{E}_*(X)$ and i_*: $\mathscr{E}_*(X) \to i_*\bar{\mathscr{E}}_*(A)$ such that the following sequence of groups and homomorphisms is exact*

$$i_*[\Sigma V, A] \xrightarrow{\lambda} \mathscr{E}_*(X) \xrightarrow{i^*} i_*\bar{\mathscr{E}}_*(A).$$

Proof. This is proved in [16, Theorem 1.3]. □

We say that a group G satisfies the *maximal condition* if G and all of its subgroups are finitely generated.

A.3. Proposition. *If $\mathscr{E}_*(A)$ satisfies the maximal conditions, then $\mathscr{E}_*(X)$ satisfies the maximal condition.*

Proof. Since i_*: $[\Sigma V, A] \to [\Sigma V, X]$ is a homomorphism of finitely generated abelian groups, $i_*[\Sigma V, A]$ satsfies the maximal condition. It suffices by Proposition A.2 to show that $i_*\bar{\mathscr{E}}_*(A)$ satisfies the maximal condition. Since $\mathscr{E}_*(A)$ satisfies the maximal condition, so does $\bar{\mathscr{E}}_*(A) \subseteq \mathscr{E}_*(A)$. However, $i_*: \bar{\mathscr{E}}_*(A) \to i_*\bar{\mathscr{E}}_*(A)$ is an epimorphism, and so $i_*\bar{\mathscr{E}}_*(A)$ satisfies the maximal condition. □

A.4. Proposition. *If Y is a 1-connected, finite complex, then $\mathscr{E}_*(Y)$ satisfies the maximal condition. In particular, $\mathscr{E}_*(Y)$ is finitely generated.*

Proof. We sketch the proof which proceeds inductively over the skeleta Y^n of Y. For the inductive step from Y^{n-1} to Y^n we apply Proposition A.3 with $A = Y^{n-1}$, f: $V = S_1^{n-1} \vee \cdots \vee S_r^{n-1} \to A$ the map which attaches n-cells to A to form Y^n and $X = Y^n$. The existence of $V_1 \subseteq V$ such that $\ker f_* = H_{n-1}(S_{m+1}^{n-1} \vee \cdots \vee S_r^{n-1})$ is assured by [16, Lemma 2.1]. This completes the sketch of the proof. □

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